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SEISMIC WAVE PROPAGATION IN A SELF-GRAVITATING ANISOTROPIC EARTH

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CONTENTS

The mathematical theory is discussed of the propagation of acceleration waves in a spherical, elastic Earth, which has spherical symmetry and the material of which is transversely isotropic at each point with respect to the radial direction, both as a result of its intrinsic nature and as a result of the effect of self-gravitation. An arbitrary variation of density with radial distance from the Earth's centre is assumed. The differential equation for the ray path is obtained and solved in the case of an *SH*-wave. It is seen that the usual Herglotz–Wiechert method for the determination of the dependence of wave speed on radial position breaks down.

1. Introduction

The mathematical theory of the propagation of seismic waves in the Earth is generally based on the assumption that the material of the Earth is isotropic and that the stress at every point in it, due to self-gravitation, is a hydrostatic pressure. The constitutive equation relating the stresses and strains associated with seismic waves is assumed to be the usual constitutive equation of classical elasticity theory for isotropic materials. Seismological observations are usually interpreted in terms of a theory of this type.

Stoneley (1949) appears to have been the first geophysicist to consider the seismological implications of anisotropy. He concluded that quite erroneous conclusions could be reached if data obtained from seismic waves propagating in a transversely isotropic layer are interpreted in terms of a mathematical model in which this transverse isotropy is not properly taken into account. More recently, several investigators (e.g. Anderson (1961) and Backus (1962)) have discussed the propagation of elastic waves in layered transversely isotropic media. The question of whether anisotropy needs to be assumed in order to account adequately for the observed behaviour of seismic waves has also been considered, in general inconclusively. Some of these investigations have been described by Nuttli (1963), who pointed out that the uncertainty of the conclusions may result from scatter in the experimental data and from the particular method used in the reduction of the data. In all of these investigations it was assumed that the axis of rotational symmetry of the transversely isotropic material has the same direction at all points.

Helbig (1966) has considered the propagation of seismic waves in a spherical Earth, which has spherical symmetry, and the material of which is transversely isotropic at each point with respect to the radial direction at the point. In studying this problem he uses a geometrical ray-tracing procedure.

In the present paper, we discuss the mathematical theory of the propagation of acceleration waves in a spherical, elastic Earth, which has spherical symmetry and the material of which is transversely isotropic at each point with respect to the radial direction, both as a result of its intrinsic nature and as a result of the effect of self-gravitation. An arbitrary variation of density with radial distance from the centre of the Earth is assumed. The seismic waves then propagate in a medium in which there exists a spherically symmetric state of initial stress, the initial stress at each point having transverse isotropy with respect to the radial direction at the point. This state of initial stress resulting from self-gravitation is fully discussed in §§ 2 and 3. The relations between the increment in the stress and the displacement gradients due to the propagation of a seismic wave are then obtained by systematic linearization of the constitutive equations for a transversely isotropic elastic material subjected to a small deformation superposed on a finite deformation (see § 4). Using these expressions for the stress increment, equations of motion are obtained in terms of the small superposed displacements.

In § 5, we obtain the secular equation for an acceleration wave of finite amplitude propagating in the Earth, using techniques introduced by Varley (1965) and Varley & Cumberbatch (1965), and following to some extent the paper by Bland (1965). The simplifications resulting from the assumptions that the wave amplitude is small are introduced in § 6 and the resulting secular equation is solved, yielding, in general, three wave speeds for any specified inclination of the wave normal to the radial direction at the point considered. One of these corresponds to a transverse wave perpendicular to the plane of propagation (*SH*-wave), while the other two are, in general, neither longitudinal nor transverse. It is found in § 7 that there are generally five principal wave speeds—two corresponding to longitudinal waves and three to transverse waves—in agreement with the analysis of Green (1963).

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In §§ 8–10, we discuss the passage of a ray through the Earth. Only the *SH*-wave is considered. In general, the other waves do not propagate unchanged in form. In § 8 we derive differential equations for the path of the acceleration wave in the form of expressions for the time derivatives of the vector position of the discontinuity and of the unit normal to the wave front. From these results, we obtain alternative differential equations for the path in the form of expressions for the time derivatives of the radial distance R of the discontinuity from the centre of the Earth and of the inclination θ of the radial direction at the point to a fixed radial direction.

In § 9, the equations of § 8 are used to obtain explicit expressions for the time of travel of the discontinuity and the change Δ in the value of θ for specified change of R. In § 10, these results are used to derive an expression for the derivative of the time of travel of the wave between two points on the Earth's surface with respect to the angle subtended at the Earth's centre by these points. It is found that this derivative is the quotient of the radial distance from the Earth's centre to the point of deepest penetration of the ray and the speed of the wave at this point, paralleling a result obtained in the classical case of an isotropic Earth. It is seen, however, that the classical Herglotz–Wiechert method for the determination of the dependence of wave-speed on radial distance from the Earth's centre generally breaks down, in the case when the material of the Earth is assumed to be intrinsically anisotropic, or when the stresses resulting from self-gravitation are anisotropic. Thus, use of the Herglotz–Wiechert method will lead to erroneous results for the dependence of wave speed on radial position and possibly to erroneous deductions from this dependence regarding the constitution of the Earth.

2. The equilibrium configuration

We consider a sphere of radius R_0 which has a spherically-symmetric mass density distribution, $D = D(R)$, and which is in equilibrium with self-gravitational body forces. The equilibrium stress is assumed to be radially- and spherically-symmetric. That is,

$$\left.\begin{aligned}\text{radial stress} &= \Sigma(R),\\ \text{hoop stress} &= \sigma(R),\end{aligned}\right\} \tag{2·1}$$

where, in general, $\Sigma \neq \sigma$. We denote this equilibrium configuration as B.

Let X be a fixed rectangular frame with origin O at the centre of the sphere. Let X_A denote the coordinates of a generic particle of B measured in X. Then the radial distance R is $(X_A X_A)^{\frac{1}{2}}$.

With (2·1) the components of the Cauchy stress tensor relative to X are

$$S_{AB} = \sigma\delta_{AB} + (\Sigma - \sigma)\, J_A J_B, \tag{2·2}$$

where

$$J_A = X_A/R \tag{2·3}$$

is the unit radial vector.

The body force acting in B is taken as the usual Newtonian force. Thus, since the total mass within a sphere of radius R is

$$4\pi \int_0^R D(Z)\, Z^2\, \mathrm{d}Z, \tag{2·4}$$

the magnitude of the force per unit mass at this radius is

$$G(R) = 4\pi\gamma R^{-2}\int_0^R D(Z)\,Z^2\,\mathrm{d}Z, \tag{2·5}$$

where γ is the gravitational constant. The body force vector (per unit mass) is then

$$F_A = -J_A G, \tag{2·6}$$

where J_A is given by (2·3).

The equations of equilibrium are†

$$S_{AB,B} + DF_A = 0. \tag{2·7}$$

Substitution from (2·2) and (2·6) into (2·7) yields the single equation of equilibrium‡

$$\Sigma' + \frac{2}{R}(\Sigma - \sigma) = DG. \tag{2·8}$$

We assume the surface traction vector vanishes on the outer surface of the sphere B. This requires

$$[S_{AB}J_B]_{R=R_0} = 0, \tag{2·9}$$

and, with (2·2), (2·9) yields

$$\Sigma(R_0) = 0. \tag{2·10}$$

In the following pages Σ and σ are assumed to satisfy equations (2·8) and (2·10).

3. Equations governing finite deformations

In a deformation from the equilibrium state B the particle X_A occupies position x_i, measured in X, at time t and we write

$$x_i = x_i(X_A, t). \tag{3·1}$$

In configuration B the material is assumed to be elastic and transversely isotropic with respect to the radial direction. Material properties also depend upon radial distance. Thus, the strain energy, W, measured per unit volume of B, is of the form

$$W = W(x_{i,A}, J_A, R), \tag{3·2}$$

where $x_{i,A}$ is the deformation gradient computed from (3·1), J_A is given by (2·3) and R is $(X_A X_A)^{\frac{1}{2}}$. With (3·2) the Piola–Kirchhoff stress tensor is given by§

$$\mathfrak{s}_{Ai} = \partial W/\partial x_{i,A}. \tag{3·3}$$

The strain energy must remain unchanged if the material undergoes a rigid-body motion. This requirement is satisfied if W is taken to depend on $x_{i,A}$ through the Cauchy–Green strain tensor.

$$G_{AB} = x_{i,A}x_{i,B}. \tag{3·4}$$

In place of (3·2) we write

$$W = W(G_{AB}, J_A, R), \tag{3·5}$$

and, with (3·4), (3·3) becomes

$$\mathfrak{s}_{Ai} = 2x_{i,K}\,\partial W/\partial G_{AK}, \tag{3·6}$$

in which W is assumed to be written as a symmetric function|| of G_{AB}.

† The notation $_{,A}$ is used to denote the operator $\partial/\partial X_A$.

‡ Throughout we use prime to denote differentiation with respect to R.

§ In terms of $\mathfrak{s}_{Ai}$ the Cauchy stress tensor is $s_{ji} = x_{j,A}\,\mathfrak{s}_{Ai}|x_{k,K}|^{-1}$.

|| That is, $\partial W/\partial G_{AB} = \partial W/\partial G_{BA}$.

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In appendix 1 it is shown that W depends on G_{AB} and J_A through the five invariants $I_1, \ldots, I_5$ given by

$$\left.\begin{aligned} I_1 = G_{AA}, \quad I_2 = \tfrac{1}{2}(G_{AA}G_{BB} - G_{AB}G_{AB}), \quad I_3 = |G_{AB}|, \\ I_4 = J_A J_B G_{AB} \quad \text{and} \quad I_5 = J_A J_B G_{AK} G_{KB}. \end{aligned}\right\} \tag{3·7}$$

Thus, in place of (3·5) we write

$$W = W(I_\alpha, R) \quad (\alpha = 1, \ldots, 5). \tag{3·8}$$

With the form (3·8) for W, it follows that†

$$\frac{\partial W}{\partial G_{AK}} = \frac{\partial W}{\partial I_\alpha} \frac{\partial I_\alpha}{\partial G_{AK}}, \tag{3·9}$$

where, with the aid of (3·7) we have

$$\left.\begin{aligned} &\frac{\partial I_1}{\partial G_{AK}} = \delta_{AK}, \quad \frac{\partial I_2}{\partial G_{AK}} = I_1 \delta_{AK} - G_{AK}, \\ &\frac{\partial I_3}{\partial G_{AK}} = \tfrac{1}{2}\epsilon_{APQ}\epsilon_{KLM} G_{PL} G_{QM}, \\ &\frac{\partial I_4}{\partial G_{AK}} = J_A J_K \quad \text{and} \quad \frac{\partial I_5}{\partial G_{AK}} = (G_{AP} J_K + G_{KP} J_A) J_P. \end{aligned}\right\} \tag{3·10}$$

Substitution from (3·9) and (3·10) into (3·6) yields

$$\begin{aligned} \mathfrak{z}_{Ai} = 2x_{i,K}\bigg\{\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\delta_{AK} - \frac{\partial W}{\partial I_2} G_{AK} + \frac{1}{2}\frac{\partial W}{\partial I_3}\epsilon_{APQ}\epsilon_{KLM} G_{PL} G_{QM} \\ + \frac{\partial W}{\partial I_4} J_A J_K + \frac{\partial W}{\partial I_5}(G_{AP} J_K + G_{KP} J_A) J_P\bigg\}. \end{aligned} \tag{3·11}$$

The form of the dependence of W in (3·8) upon the I_α is not entirely arbitrary. It is restricted by the requirement that the stress components (3·11) coincide with (2·2) when the body is in the equilibrium state B. In this case (cf. (3·1)) $x_{i,A} = \delta_{iA}$ and (3·4) and (3·7) become

$$\left.\begin{aligned} G_{AB} = \delta_{AB}, \quad I_1 = I_2 = 3, \\ I_3 = I_4 = I_5 = 1. \end{aligned}\right\} \tag{3·12}$$

With (3·12) and the notation (cf. (3·8))

$$W_\alpha = \frac{\partial W}{\partial I_\alpha}(3, 3, 1, 1, 1, R), \tag{3·13}$$

equation (3·11) becomes

$$\mathfrak{z}_{Ai} = 2\delta_{iK}\{(W_1 + 2W_2 + W_3)\delta_{AK} + (W_4 + 2W_5) J_A J_K\}. \tag{3·14}$$

A comparison of (3·14) and (2·2) shows that

$$\left.\begin{aligned} 2(W_1 + 2W_2 + W_3) = \sigma, \\ 2(W_4 + 2W_5) = \Sigma - \sigma. \end{aligned}\right\} \tag{3·15}$$

From (3·15),

$$\Sigma = 2(W_1 + 2W_2 + W_3 + W_4 + 2W_5). \tag{3·16}$$

† A repeated lower-case Greek subscript indicates summation over 1, …, 5.

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Substitution from $(3\cdot15)_2$ and (3·16) into (2·8) and (2·10) gives restrictions on the dependence of W upon R in the configuration B.

In a deformation of the self-gravitating body, the body force changes with time. For Newtonian (inverse-square) attraction the body force vector, per unit mass, at x_i at time t is

$$f_i = \gamma \int_B \frac{D(Z)\,(z_i - x_i)}{\{(z_m - x_m)\,(z_m - x_m)\}^{\frac{3}{2}}}\,\mathrm{d}Z_1\,\mathrm{d}Z_2\,\mathrm{d}Z_3, \tag{3·17}$$

where (cf. (3·1))

$$\left.\begin{aligned} x_i &= x_i(X_A, t), \\ z_i &= x_i(Z_A, t), \end{aligned}\right\} \tag{3·18}$$

and where we have written $Z = (Z_K Z_K)^{\frac{1}{2}}$ for the argument of D. The integration in (3·17) is carried out in the fixed spherical configuration B.

The Piola–Kirchhoff equations of motion are†

$$\tilde{s}_{Ai,A} + Df_i = D\ddot{x}_i. \tag{3·19}$$

Substitution from (3·11) and (3·17) into (3·19) yields a differential–integral equation for the deformation (3·1).

Let K denote a surface in B with unit normal vector N_A. Under the deformation (3·1) K becomes the surface k_t at time t. The surface traction vector acting across k_t, measured per unit area of K, is

$$\tilde{t}_i = \tilde{s}_{Ai} N_A, \tag{3·20}$$

where $\tilde{s}_{Ai}$ is given by (3·11). The rate at which $\tilde{t}_i$ does work, per unit area of K, is $\tilde{t}_i \dot{x}_i$. We define the *material energy flux vector* by

$$\tilde{p}_A = -\tilde{s}_{Ai} \dot{x}_i. \tag{3·21}$$

By (3·21) and (3·20), $-\tilde{p}_A N_A$ is the rate of energy flow at time t per unit area of a surface whose unit normal vector in B is N_A.

4. Equations governing small deformations

To obtain the equations appropriate to infinitesimal deformations from the equilibrium state B, we systematically linearize the appropriate equations of § 3. This is conveniently done by rewriting (3·1) in the form

$$x_B = X_B + \epsilon u_B(X_A, t), \tag{4·1}$$

where ϵ is regarded as a small constant whose square may be uniformly neglected. In this section equations will be written correct to the first order in ϵ without specific mention in each case that higher order terms have been neglected.

Using (4·1) in (3·4) we have

$$G_{AB} = \delta_{AB} + \epsilon(u_{A,B} + u_{B,A}). \tag{4·2}$$

We write

$$I_\alpha = I^0_\alpha + \epsilon \bar{I}_\alpha \quad (\alpha = 1, \ldots, 5), \tag{4·3}$$

where (cf. (3·12))

$$I^0_\alpha = \left\{\begin{matrix} 3 & (\alpha = 1, 2), \\ 1 & (\alpha = 3, 4, 5). \end{matrix}\right\} \tag{4·4}$$

† A superposed dot denotes the operator $\partial/\partial t$.

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Substitution from (4·2) into (3·7) yields, with (4·4),

$$\left.\begin{aligned}\bar{I}_1 = 2u_{A,A}, \quad \bar{I}_2 = 4u_{A,A}, \quad \bar{I}_3 = 2u_{A,A}, \\ \bar{I}_4 = 2J_A J_B u_{A,B} \quad \text{and} \quad \bar{I}_5 = 4J_A J_B u_{A,B}.\end{aligned}\right\} \tag{4·5}$$

We introduce the notation (cf. (3·13))

$$W_{\alpha\beta} = W_{\beta\alpha} = \frac{\partial^2 W}{\partial I_\alpha \, \partial I_\beta}(3,\,3,\,1,\,1,\,1,\,R) \quad (\alpha,\beta = 1, \ldots, 5) \tag{4·6}$$

and write
$$\partial W/\partial I_\alpha = W_\alpha + \epsilon W_{\alpha\beta}\bar{I}_\beta, \tag{4·7}$$
where W_α is given by (3·13).

For use in the stress equation (3·11) the following quantities are computed by using (4·1) to (4·7):

$$\begin{aligned}
2x_{B,K}\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\delta_{AK} &= 2(W_1+3W_2)\,\delta_{AB} + 2\epsilon(W_1+3W_2)\,\delta_{AL}\delta_{BK}u_{K,L} \\
&\quad + 4\epsilon\{\delta_{AB}\delta_{KL}(W_2+W_{11}+5W_{12}+W_{13}+6W_{22}+3W_{23}) \\
&\quad + \delta_{AB}J_K J_L(W_{14}+2W_{15}+3W_{24}+6W_{25})\}\,u_{K,L}, \\
2x_{B,K}\frac{\partial W}{\partial I_2}G_{AK} &= 2W_2\delta_{AB} + 2\epsilon W_2\delta_{AL}\delta_{BK}u_{K,L} + 2\epsilon W_2(\delta_{AK}\delta_{BL}+\delta_{AL}\delta_{BK})\,u_{K,L} \\
&\quad + 4\epsilon\{\delta_{AB}\delta_{KL}(W_{12}+2W_{22}+W_{23}) + \delta_{AB}J_K J_L(W_{24}+2W_{25})\}\,u_{K,L}, \\
x_{B,K}\frac{\partial W}{\partial I_3}\epsilon_{APQ}\epsilon_{KLM}G_{PL}G_{QM} &= 2W_3\delta_{AB} + 2\epsilon W_3\delta_{AL}\delta_{BK}u_{K,L} - 2\epsilon W_3(\delta_{AK}\delta_{BL}+\delta_{AL}\delta_{BK}) \\
&\quad + 4\epsilon\{\delta_{AB}\delta_{KL}(W_3+W_{13}+2W_{23}+W_{33}) \\
&\quad + \delta_{AB}J_K J_L(W_{34}+2W_{35})\}\,u_{K,L}, \\
2x_{B,K}\frac{\partial W}{\partial I_4}J_A J_K &= 2W_4 J_A J_B + 2\epsilon W_4 J_A J_L\delta_{BK}u_{K,L} \\
&\quad + 4\epsilon\{J_A J_B\delta_{KL}(W_{14}+2W_{24}+W_{34}) + J_A J_B J_K J_L(W_{44}+2W_{45})\}\,u_{K,L}, \\
2x_{B,K}\frac{\partial W}{\partial I_5}(G_{AP}J_K + G_{KP}J_A)\,J_P &= 4W_5 J_A J_B + 4\epsilon W_5 J_A J_L\delta_{BK}u_{K,L} \\
&\quad + 2\epsilon W_5\{\delta_{AK}J_B J_L + J_A J_K\delta_{BL} + \delta_{AL}J_B J_K + J_A J_L\delta_{BK}\}\,u_{K,L} \\
&\quad + 8\epsilon\{J_A J_B\delta_{KL}(W_{15}+2W_{25}+W_{35}) \\
&\quad + J_A J_B J_K J_L(W_{45}+2W_{55})\}\,u_{K,L}.
\end{aligned} \tag{4·8}$$

Substitution from (4·8) into (3·11) and use of (3·14) yields

$$\mathfrak{z}_{AB} = S_{AB} + \epsilon(S_{AL}\delta_{BK} + C_{ABKL})\,u_{K,L}, \tag{4·9}$$

where S_{AB} is given by (2·2) and where

$$\begin{aligned}C_{ABKL} &= C_1\delta_{AB}\delta_{KL} + C_2(\delta_{AK}\delta_{BL}+\delta_{AL}\delta_{BK}) + C_3(\delta_{AB}J_K J_L + J_A J_B\delta_{KL}) \\ &\quad + C_4(\delta_{AK}J_B J_L + J_A J_K\delta_{BL} + \delta_{AL}J_B J_K + J_A J_L\delta_{BK}) + C_5 J_A J_B J_K J_L,\end{aligned} \tag{4·10}$$

in which the C_α are functions of R defined by

$$\left.\begin{aligned}&C_1 = 4(W_2+W_3+W_{11}+4W_{12}+2W_{13}+4W_{22}+4W_{23}+W_{33}), \\ &C_2 = -2(W_2+W_3), \quad C_3 = 4(W_{14}+2W_{15}+2W_{24}+4W_{25}+W_{34}+2W_{35}), \\ &C_4 = 2W_5 \quad \text{and} \quad C_5 = 4(W_{44}+4W_{45}+4W_{55}).\end{aligned}\right\} \tag{4·11}$$

From (4·10), C_{ABKL} has the symmetries of the classical elasticity tensor; i.e.

$$C_{ABKL} = C_{BAKL} = C_{ABLK} = C_{KLAB}. \tag{4·12}$$

The appropriate expression for the body force is obtained by replacing (3·18) by (cf. (4·1))

$$\left.\begin{aligned} x_B &= X_B + \epsilon u_B(X_K, t), \\ z_B &= Z_B + \epsilon u_B(Z_K, t), \end{aligned}\right\} \tag{4·13}$$

and by substituting from (4·13) into (3·17). This yields

$$f_B = F_B + \epsilon \bar{F}_B, \tag{4·14}$$

where (cf. (2·6))

$$\left.\begin{aligned} F_B &= \gamma \int_B \frac{D(Z)\,(Z_B - X_B)}{\{(Z_L - X_L)\,(Z_L - X_L)\}^{\frac{3}{2}}}\,\mathrm{d}Z_1\,\mathrm{d}Z_2\,\mathrm{d}Z_3 = -J_B G, \\ \bar{F}_B &= \gamma \int_B \frac{D(Z)\,\{u_A(Z_K, t) - u_A(X_K, t)\}}{\{(Z_L - X_L)\,(Z_L - X_L)\}^{\frac{3}{2}}} \left\{\delta_{AB} - 3\,\frac{(Z_A - X_A)\,(Z_B - X_B)}{(Z_M - X_M)\,(Z_M - X_M)}\right\} \mathrm{d}Z_1\,\mathrm{d}Z_2\,\mathrm{d}Z_3. \end{aligned}\right\} \tag{4·15}$$

Substitution from (4·1), (4·9) and (4·14) into the equations of motion (3·19) yields

$$S_{AB,A} + \epsilon(S_{AL}\delta_{BK} + C_{ABKL})\,u_{K,LA} + \epsilon(S_{AL}\delta_{BK} + C_{ABKL})_{,A}\,u_{K,L} + DF_B + \epsilon D\bar{F}_B = \epsilon D\ddot{u}_B,$$

or, since (cf. (2·7)) $S_{AB,A} + DF_B = 0$, this becomes

$$(S_{AL}\delta_{BK} + C_{ABKL})\,u_{K,LA} + (S_{AL}\delta_{BK} + C_{ABKL})_{,A}\,u_{K,L} + D\bar{F}_B = D\ddot{u}_B. \tag{4·16}$$

These are the equations of motion governing infinitesimal deformations from the equilibrium state B. In (4·16), S_{AL} is given by (2·2), C_{ABKL} by (4·10) and $\bar{F}_B$ by $(4·15)_2$.

From (3·21), with (4·1), the material energy flux vector is

$$\tilde{p}_A = -\epsilon \mathfrak{F}_{AB}\,\dot{u}_B. \tag{4·17}$$

Substitution from (4·9) into (4·17) yields

$$\tilde{p}_A = -\epsilon S_{AB}\,\dot{u}_B - \epsilon^2(S_{AL}\delta_{BK} + C_{ABKL})\,u_{K,L}\,\dot{u}_B. \tag{4·18}$$

5. Finite acceleration waves

Across an acceleration wave front the acceleration $\ddot{x}_i$ suffers a jump discontinuity as the particle X_A is traversed by the front while (cf. (3·1)) x_i, $\dot{x}_i$ and $x_{i,A}$ remain continuous.

Following Varley (1965) we replace (X_A, t) by new independent variables (X_A, ϕ) where $\phi = \Phi(X_A, t)$ is continuously differentiable and such that $\phi = 0$ yields the material description of an isolated acceleration wave front.† The unit vector normal to $\phi =$ constant is

$$N_A = \Phi_{,A}/(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}}, \tag{5·1}$$

and the speed of propagation of this surface, relative to material in B, is given by

$$V = -\dot{\Phi}/(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}}. \tag{5·2}$$

Throughout this section we assume that $\dot{\Phi}$ (and, hence, V) is not zero.

† At time t, the locus $\Phi(X_A, t) = 0$ is a smooth surface in the reference configuration B. Letting X_A^+ and X_A^- denote points on opposite sides of this surface, then for all X_A on $\Phi(X_A, t) = 0$, $\ddot{x}_i(X_A^+, t)$ and $\ddot{x}_i(X_A^-, t)$ approach definite limits as $X_A^+ \to X_A$ and $X_A^- \to X_A$ but these limiting values of $\ddot{x}_i$ are not equal.

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For any function $f(X_A, t)$ we define $F(X_A, \phi)$ by

$$F(X_A, \phi) = f(X_A, t(X_A, \phi)), \tag{5·3}$$

where $t(X_A, \phi)$ is defined implicitly by the relationship $\phi = \Phi(X_A, t)$. On a surface $\phi =$ constant,

$$\Phi_{,B} + \dot{\Phi}\frac{\partial t}{\partial X_B} = 0,$$

or using (5·1) and (5·2),

$$\frac{\partial t}{\partial X_B} = \frac{N_B}{V}. \tag{5·4}$$

Differentiating (5·3) with respect to X_B and using (5·4) yields

$$F_{,B} = f_{,B} + \dot{f}N_B/V. \tag{5·5}$$

It is convenient to introduce functions p_{iA} and v_i for a motion $x_i(X_A, t)$ by the definitions

$$\left.\begin{aligned} p_{iA} &= x_{i,A}, \\ v_i &= \dot{x}_i. \end{aligned}\right\} \tag{5·6}$$

These quantities satisfy the compatibility relation

$$\dot{p}_{iA} = v_{i,A}. \tag{5·7}$$

The counterparts of (5·3) for p_{iA} and v_i are P_{iA} and V_i, respectively. Thus, on a surface $\phi =$ constant we have, using (5·5),

$$\left.\begin{aligned} P_{iA,B} &= p_{iA,B} + \dot{p}_{iA}N_B/V, \\ V_{i,A} &= v_{i,A} + \dot{v}_i N_A/V. \end{aligned}\right\} \tag{5·8}$$

At the acceleration front, $\phi = 0$, we note that P_{iA} and V_i, regarded as functions of X_A, are continuous. Further, since $P_{iA,B}$ and $V_{i,A}$ are derivatives interior (tangential) to the wave front, they are continuous at $\phi = 0$. We denote the jump in a function $f(X_A, t)$ as X_A is traversed by the wave at time t by $[f]$. From (5·7) and (5·8) follow

$$\left.\begin{aligned} [\dot{p}_{iA}] &= [v_{i,A}], \\ [p_{iA,B}] + [\dot{p}_{iA}]N_B/V &= 0, \\ [v_{i,A}] + [\dot{v}_i]N_A/V &= 0. \end{aligned}\right\} \tag{5·9}$$

These equations yield

$$[p_{iA,B}] = [\dot{v}_i]N_A N_B/V^2,$$

or, with (5·6), we obtain the known result,

$$[x_{i,AB}] = [x_{i,BA}] = [\ddot{x}_i]N_A N_B/V^2. \tag{5·10}$$

At an acceleration front the equations of motion (3·19) yield

$$[\mathfrak{z}_{Ai,A}] + [Df_i] = [D\ddot{x}_i]. \tag{5·11}$$

From (3·3), with W given by (3·2), we have

$$\mathfrak{z}_{Ai,A} = \frac{\partial^2 W}{\partial x_{i,A}\partial x_{k,B}}x_{k,BA} + \frac{\partial^2 W}{\partial x_{i,A}\partial J_B}\frac{\partial J_B}{\partial X_A} + \frac{\partial^2 W}{\partial x_{i,A}\partial R}\frac{\partial R}{\partial X_A},$$

so that

$$[\mathfrak{z}_{Ai,A}] = \frac{\partial^2 W}{\partial x_{i,A}\partial x_{k,B}}[x_{k,BA}], \tag{5·12}$$

where it is assumed that W has continuous second derivatives. Assuming that the mass density D is continuous, it follows from (3·17) that f_i is continuous so that $[Df_i] = 0$. Therefore, with (5·12), (5·11) becomes

$$\frac{\partial^2 W}{\partial x_{i,A}\,\partial x_{k,B}}[x_{k,BA}] = D[\ddot{x}_i]. \tag{5·13}$$

Using (5·10), (5·13) yields

$$\left\{\frac{\partial^2 W}{\partial x_{i,A}\,\partial x_{k,B}} N_A N_B - DV^2\delta_{ik}\right\}[\ddot{x}_k] = 0. \tag{5·14}$$

Equation (5·14) is the material form of the propagation condition for an acceleration wave. According to this, the amplitude $[\ddot{x}_i]$ must be an eigen-vector of the (symmetric) *material acoustic tensor*

$$\tilde{q}_{ik} = \frac{\partial^2 W}{\partial x_{i,A}\,\partial x_{k,B}} N_A N_B. \tag{5·15}$$

The speed of propagation, V, must be such that DV^2 is the corresponding eigen-value. From (5·14), possible speeds of propagation of acceleration waves are solutions of the secular equation

$$|\tilde{q}_{ik} - DV^2\delta_{ik}| = 0, \tag{5·16}$$

where $\tilde{q}_{ik}$ is given by (5·15). We note that $[\ddot{x}_i]$ and V depend on the (material) direction of propagation† N_A.

Material immediately ahead of a wave front may be undergoing any smooth deformation which is consistent with the equations of motion (3·19). In particular, suppose the material has always been in configuration B prior to time $t = 0$. At $t = 0$ imagine that some portion of the outer surface of the body is suddenly accelerated. In principle this would give rise to an acceleration wave propagating into the interior of B. Motion of material particles behind the wave front would alter the body force, instantaneously, throughout the entire body in accord with (3·17). Hence, the material particle X_A would generally be in motion prior to the arrival of the front at time t. That is, due to the presence of self-gravitating body forces, we would generally expect that (cf. (3·1))

$$x_i \neq \delta_{iA}X_A, \quad \dot{x}_i \neq 0 \quad \text{and} \quad x_{i,A} \neq \delta_{iA} \tag{5·17}$$

at the wave front. If inequality $(5·17)_3$ holds it follows that the stress (3·11) is not identical with (2·2) at the front. It is apparent that an analysis of the effects due to self-gravitational body forces on conditions which exist at a wave front could be made only after an acceleration wave solution had been established. We shall not attempt such an analysis in the present paper.

† The relation of (5·15) and (5·16) to corresponding equations given by Truesdell (1961) is furnished by the correspondence between spatial and material representations of propagating surfaces. In the spatial representation a wave front has unit normal n_i and local speed of propagation v where (see appendix 2, equation (12·10)) $N_A/V = n_j x_{j,A}/v$. Thus, using (5·15), we have

$$\tilde{q}_{ik}/V^2 = q_{ik}/v^2$$

where

$$q_{ik} = \frac{\partial^2 W}{\partial x_{i,A}\,\partial x_{k,B}} x_{j,A}x_{l,B}n_j n_l$$

is the local acoustic tensor. The counterpart of (5·16) is

$$|q_{ik} - Dv^2\delta_{ik}| = 0.$$

As has been pointed out by Truesdell (1961), body forces do not affect the propagation condition (5·14). Whatever the value of $x_{i,A}$ may be, the amplitude $[\ddot{x}_i]$ of an acceleration wave which propagates in the (material) direction N_A must be an eigenvector of $\tilde{q}_{ik}$ with (cf. (5·15)) $\partial^2 W/\partial x_{i,A}\,\partial x_{k,B}$ evaluated at $x_{i,A}$.

If the form (3·5) for W is used an alternative expression for $\tilde{q}_{ik}$ is obtained. From (3·5), with (3·4), we have

$$\frac{\partial^2 W}{\partial x_{i,A}\,\partial x_{k,B}} = 2\delta_{ik}\frac{\partial^2 W}{\partial G_{AB}} + 4x_{i,P}\,x_{k,Q}\,\frac{\partial^2 W}{\partial G_{AP}\,\partial G_{BQ}}. \tag{5·18}$$

Using (5·18), we obtain from (5·15)

$$\tilde{q}_{ik} = \left\{2\delta_{ik}\frac{\partial W}{\partial G_{AB}} + 4x_{i,P}\,x_{k,Q}\,\frac{\partial^2 W}{\partial G_{AP}\,\partial G_{BQ}}\right\} N_A N_B. \tag{5·19}$$

Equation (5·19) does not reflect explicitly the fact that the material of B is transversely isotropic. We can obtain an expression for $\tilde{q}_{ik}$ which does so by substituting in (5·19) the expression (3·5) for W. From (3·5), we have

$$\left.\begin{aligned} \frac{\partial W}{\partial G_{AB}} &= \frac{\partial W}{\partial I_\alpha}\frac{\partial I_\alpha}{\partial G_{AB}},\\ \frac{\partial^2 W}{\partial G_{AP}\,\partial G_{BQ}} &= \frac{\partial^2 W}{\partial I_\alpha\,\partial I_\beta}\frac{\partial I_\alpha}{\partial G_{AP}}\frac{\partial I_\beta}{\partial G_{BQ}} + \frac{\partial W}{\partial I_\alpha}\frac{\partial^2 I_\alpha}{\partial G_{AP}\,\partial G_{BQ}}. \end{aligned}\right\} \tag{5·20}$$

The form for $\tilde{q}_{ik}$ obtained by expanding the derivatives appearing in (5·20) and substituting into (5·19) will not be displayed here.

If, at a particle X_A, the instantaneous value of $x_{i,A}$ is δ_{iA} as a wave passes, then conditions (3·12) hold. It follows that the stress is (cf. (3·6) and (2·2))

$$S_{AB} = 2\left.\frac{\partial W}{\partial G_{AB}}\right|_{G_{AB}=\delta_{AB}}. \tag{5·21}$$

In addition, it is easily shown that

$$4\left.\frac{\partial^2 W}{\partial G_{AP}\,\partial G_{BQ}}\right|_{G_{AB}=\delta_{AB}} = C_{APBQ}, \tag{5·22}$$

where C_{APBQ} is given by (4·10). From (5·19), with (5·21) and (5·22), we have

$$\tilde{q}_{ik} = \{\delta_{ik}S_{AB} + \delta_{iP}\delta_{kQ}C_{APBQ}\}\,N_A N_B, \tag{5·23}$$

which is identical with the linear acoustic tensor to be derived in the following section (cf. (6·5)). Hence, in the special case when material at a wave front is in the state of stress S_{AB} (i.e. when $x_{i,A} = \delta_{iA}$) the propagation condition for finite acceleration waves is equivalent to that obtained from linear theory.

6. The secular equation for seismic waves and its solutions

The usefulness of linear elasticity theory in describing seismic wave transmission has been well established. In this section we shall combine results from §§ 4 and 5 to obtain the speeds of propagation of seismic waves for the body B.

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We assume the motion (4·1) is that associated with the propagation of an acceleration wave. That is, u_K, $\dot{u}_K$ and $u_{K,L}$ are continuous throughout B for all time t while the acceleration $\ddot{u}_K$ suffers a jump discontinuity as the wave front traverses the particle X_A at time t. At points other than on a wave front, (4·1) yields

$$x_{K,LA} = \epsilon u_{K,LA} \quad \text{and} \quad \ddot{x}_K = \epsilon \ddot{u}_K. \tag{6·1}$$

At points on a wave front, (6·1) and (5·10) show that discontinuities in second derivatives of the displacement satisfy

$$[u_{K,LA}] = [\ddot{u}_K]\, N_L N_A / V^2, \tag{6·2}$$

where N_A and V have the same meanings as in § 5; i.e. N_A is the material unit normal vector to the wave front and V is the speed of propagation relative to material in B.

Since u_K and $u_{K,L}$ are continuous, the equations of motion (4·16) yield

$$(S_{AL}\delta_{BK} + C_{ABKL})\,[u_{K,LA}] = D[\ddot{u}_B], \tag{6·3}$$

where it is assumed that the coefficient of $u_{K,L}$ in (4·16) is continuous. From (6·2) and (6·3) it follows that

$$(Q_{BK} - DV^2\delta_{BK})\,[\ddot{u}_K] = 0, \tag{6·4}$$

where

$$Q_{BK} = (S_{AL}\delta_{BK} + C_{ABKL})\, N_A N_L \tag{6·5}$$

is the linear acoustic tensor. In view of (4·12), Q_{BK} is symmetric. Substitution from (2·2) and (4·10) into (6·5) yields the explicit form for the linear acoustic tensor:

$$\begin{aligned} Q_{BK} = {} & \{(C_2+\sigma) + (C_4+\Sigma-\sigma)\,(J_A N_A)^2\}\,\delta_{BK} + (C_1+C_2)\,N_B N_K \\ & + (C_3+C_4)\,(J_A N_A)\,(N_B J_K + J_B N_K) + \{C_4 + C_5 (J_A N_A)^2\}\,J_B J_K. \end{aligned} \tag{6·6}$$

Equation (6·4) is the condition which must be satisfied by the acceleration discontinuity $[\ddot{u}_K]$. It is the linear counterpart of (5·14). Since, by assumption, $[\ddot{u}_K] \neq 0$, (6·4) yields the secular equation for seismic waves,

$$|Q_{BK} - DV^2\delta_{BK}| = 0. \tag{6·7}$$

Introducing (6·6) into (6·7) and expanding the determinant, we obtain the solutions

$$\left.\begin{aligned} DV^2 &= H_1, \\ \text{or} \quad DV^2 &= \tfrac{1}{2}\{H_1 + H_2 \pm [(H_2-H_1)^2 - 4H_3]^{\frac{1}{2}}\},\end{aligned}\right\} \tag{6·8}$$

where

$$\left.\begin{aligned} H_1 &= N_P N_Q\{\delta_{PQ}(C_3+\sigma) + J_P J_Q (C_4+\Sigma-\sigma)\}, \\ H_2 &= N_P N_Q\{\delta_{PQ}(C_1+2C_2+C_4+\sigma) + J_P J_Q(2C_3+3C_4+C_5+\Sigma-\sigma)\}, \\ H_3 &= N_K N_L N_P N_Q\{\delta_{KL} C_4 (C_1+C_2) + J_K J_L [C_5(C_1+C_2) - (C_3+C_4)^2]\}\,(\delta_{PQ} - J_P J_Q).\end{aligned}\right\} \tag{6·9}$$

Equations (6·8), with (6·9), give explicit forms for the squared speeds of propagation of seismic waves in terms of the mass density D, the initial stresses Σ and σ, the elasticities $C_1, \ldots, C_5$ and the unit normal vector N_A.

7. Principal and general seismic waves

Since the material of B is transversely isotropic with respect to the radial direction at every point, it follows that the radial direction and directions perpendicular to it are *preferred* directions with respect to the propagation of waves. We call waves which propagate along these preferred directions *principal waves*. Consequently, the unit normal vector N_A, associated with a principal wave satisfies either

$$N_A = \pm J_A \quad \text{or} \quad N_A J_A = 0. \tag{7·1}$$

It is convenient to let a_A denote the acceleration discontinuity $[\ddot{u}_A]$. With this change the propagation condition (6·4) becomes

$$(Q_{BK} - DV^2\delta_{BK})\, a_K = 0. \tag{7·2}$$

Throughout we assume that a_K is not the zero vector, i.e. $a_K a_K \neq 0$. A wave is called *longitudinal* or *transverse* according as its amplitude a_A satisfies the relations

$$a_A = (a_B N_B)\, N_A \quad \text{or} \quad a_A N_A = 0, \tag{7·3}$$

respectively.

A detailed discussion of principal and general waves is given in the following six subsections.

(i) *When* $N_A = \pm J_A$

If $N_A = \pm J_A$ equations (6·9) become

$$\left.\begin{aligned} H_1 &= C_2 + C_4 + \Sigma, \\ H_2 &= C_1 + 2C_2 + 2C_3 + 4C_4 + C_5 + \Sigma, \\ H_3 &= 0. \end{aligned}\right\} \tag{7·4}$$

From (6·8) and (7·4), the squared wave speeds are

$$\left.\begin{aligned} V^2 &= D^{-1}(C_2 + C_4 + \Sigma), \\ V^2 &= D^{-1}(C_1 + 2C_2 + 2C_3 + 4C_4 + C_5 + \Sigma), \\ V^2 &= D^{-1}(C_2 + C_4 + \Sigma). \end{aligned}\right\} \tag{7·5}$$

Introducing $N_A = \pm J_A$ in (6·6) and substituting the resulting expression for Q_{BK} into (7·2) yields

$$\{(C_2 + C_4 + \Sigma - DV^2)\, \delta_{BK} + (C_1 + C_2 + 2C_3 + 3C_4 + C_5)\, N_B N_K\}\, a_K = 0. \tag{7·6}$$

Substitution of the three values of V^2 from (7·5) into (7·6) yields, respectively,

$$\left.\begin{aligned} (C_1 + C_2 + 2C_3 + 3C_4 + C_5)\, N_B(N_K a_K^1) &= 0, \\ (C_1 + C_2 + 2C_3 + 3C_4 + C_5)\, (N_B N_K - \delta_{BK})\, a_K^2 &= 0, \\ (C_1 + C_2 + 2C_3 + 3C_4 + C_5)\, N_B(N_K a_K^3) &= 0, \end{aligned}\right\} \tag{7·7}$$

where we use superscripts 1, 2 and 3 on the amplitude vectors a_K to distinguish the three different cases.

We define V_1 and v_1 by

$$\left.\begin{aligned} V_1^2 &= D^{-1}(C_1 + 2C_2 + 2C_3 + 4C_4 + C_5 + \Sigma), \\ v_1^2 &= D^{-1}(C_2 + C_4 + \Sigma), \end{aligned}\right\} \tag{7·8}$$

and note that V_1^2 is the squared speed given by $(7·5)_2$ and that v_1^2 is the squared speed given by $(7·5)_1$ and $(7·5)_3$. From (7·8),

$$(C_1 + C_2 + 2C_3 + 3C_4 + C_5) = D(V_1^2 - v_1^2). \tag{7·9}$$

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In the sequel we assume that $V_1^2 \neq v_1^2$; i.e. with (7·9),

$$(C_1+C_2+2C_3+3C_4+C_5) \neq 0. \tag{7·10}$$

Thus $(7·7)_2$, with $(7·3)_1$, $(7·8)_1$ and (7·10), implies that V_1^2 is the squared speed of longitudinal waves which propagate in the radial direction; and $(7·7)_{1,3}$, with $(7·3)_2$, $(7·8)_2$ and (7·10), imply that v_1^2 is the squared speed of transverse waves which propagate in the radial direction.

(ii) *When* $N_A J_A = 0$

If $N_A J_A = 0$ equations (6·9) become

$$\left.\begin{aligned} H_1 &= C_2+\sigma, \\ H_2 &= C_1+2C_2+C_4+\sigma, \\ H_3 &= (C_1+C_2)\,C_4. \end{aligned}\right\} \tag{7·11}$$

From (6·8) and (7·11) the squared wave speeds are

$$\left.\begin{aligned} V^2 &= D^{-1}(C_2+\sigma), \\ V^2 &= D^{-1}(C_1+2C_2+\sigma), \\ V^2 &= D^{-1}(C_2+C_4+\sigma). \end{aligned}\right\} \tag{7·12}$$

Introducing the condition $N_A J_A = 0$ in (6·6) and substituting the resulting expression for Q_{BK} into (7·2) yields

$$\{(C_2+\sigma-DV^2)\,\delta_{BK}+(C_1+C_2)\,N_B N_K+C_4 J_B J_K\}\,a_K = 0. \tag{7·13}$$

Substitution of the three values of V^2 from (7·12) into (7·13) yields, respectively,

$$\left.\begin{aligned} \{(C_1+C_2)\,N_B N_K+C_4 J_B J_K\}\,a_K^4 &= 0, \\ \{(C_1+C_2)\,(N_B N_K-\delta_{BK})+C_4 J_B J_K\}\,a_K^5 &= 0, \\ \{C_4(J_B J_K-\delta_{BK})+(C_1+C_2)\,N_B N_K\}\,a_K^6 &= 0, \end{aligned}\right\} \tag{7·14}$$

where, again, superscripts are used on a_K to distinguish the three different cases.

We define V_2, v_2 and v_3 by

$$\left.\begin{aligned} V_2^2 &= D^{-1}(C_1+2C_2+\sigma), \\ v_2^2 &= D^{-1}(C_2+C_4+\sigma), \\ v_3^2 &= D^{-1}(C_2+\sigma), \end{aligned}\right\} \tag{7·15}$$

and note that V_2^2, v_2^2 and v_3^2 are the squared speeds given by $(7·12)_2$, $(7·12)_3$ and $(7·12)_1$, respectively. From (7·15) it follows that

$$\left.\begin{aligned} (C_1+C_2) &= D(V_2^2-v_3^2), \\ (C_1+C_2)-C_4 &= D(V_2^2-v_2^2). \end{aligned}\right\} \tag{7·16}$$

We shall assume that

$$\left.\begin{aligned} (C_1+C_2) &\neq 0, \\ (C_1+C_2)-C_4 &\neq 0, \end{aligned}\right\} \tag{7·17}$$

or, equivalently (cf. (7·16)), $V_2^2 \neq v_3^2$ and $V_2^2 \neq v_2^2$.

To solve $(7·14)_2$ for a_K^5 we multiply it by J_B and recall that $J_B N_B = 0$. Thus, with $(7·17)_2$, we deduce $J_K a_K^5 = 0$. Using this in $(7·14)_2$, together with $(7·17)_1$, we find that $a_B^5 = (a_K^5 N_K)\,N_B$.

Hence, $(7\cdot3)_1$ and $(7\cdot15)_1$ imply that V_2^2 is the squared speed of propagation of longitudinal waves which propagate in directions perpendicular to the radial direction.

To solve $(7\cdot14)_1$ and $(7\cdot14)_3$ for a_K^4 and a_K^6 we multiply those equations by N_B, recall that $N_B J_B = 0$, and find

$$\left.\begin{aligned}(C_1+C_2)\,N_K a_K^4 = 0,\\ \{(C_1+C_2)-C_4\}\,N_K a_K^6 = 0.\end{aligned}\right\} \tag{7·18}$$

Thus, by (7·18), (7·17), $(7\cdot15)_{2,\,3}$ and $(7\cdot3)_2$, it follows that v_2^2 and v_3^2 are squared speeds of propagation of transverse waves which propagate in directions perpendicular to the radial direction.

If we use the fact that (cf. (7.18)) $N_K a_K^4 = N_K a_K^6 = 0$ in $(7\cdot14)_1$ and $(7\cdot14)_3$, we obtain

$$\left.\begin{aligned}C_4 J_B(J_K a_K^4) = 0,\\ C_4(J_B J_K - \delta_{BK})\,a_K^6 = 0,\end{aligned}\right\} \tag{7·19}$$

respectively. From (7·15),

$$C_4 = D(v_2^2 - v_3^2). \tag{7·20}$$

Hence, if $C_4 \neq 0$ we have $v_2^2 \neq v_3^2$. In this case equations (7·19) yield

$$\left.\begin{aligned}J_K a_K^4 = 0,\\ a_B^6 = (J_K a_K^6)\,J_B.\end{aligned}\right\} \tag{7·21}$$

From $(7\cdot18)_1$, $(7\cdot21)_1$ and the fact that $J_A N_A = 0$, it follows that the vector a_B^4 may be written as

$$a_B^4 = (a_A^4 \epsilon_{APQ} N_P J_Q)\,\epsilon_{BKL} N_K J_L. \tag{7·22}$$

By (7·22) and $(7\cdot21)_2$, the amplitude vectors a_A^4 and a_B^6 of waves governed by speeds v_3^2 and v_2^2, respectively, are mutually perpendicular provided $C_4 \neq 0$.

(iii) *A discussion of principal waves*

In §§ 7 (i) and 7 (ii) it has been shown that (cf. (7·8) and (7·15))

$$\left.\begin{aligned}V_{\mathrm{I}}^2 &= D^{-1}(C_1+2C_2+2C_3+4C_4+C_5+\Sigma)\\ \text{and}\quad V_2^2 &= D^{-1}(C_1+2C_2+\sigma)\end{aligned}\right\} \tag{7·23}$$

are squared speeds of principal longitudinal waves† and that

$$\left.\begin{aligned}v_1^2 &= D^{-1}(C_2+C_4+\Sigma),\\ v_2^2 &= D^{-1}(C_2+C_4+\sigma)\\ \text{and}\quad v_3^2 &= D^{-1}(C_2+\sigma)\end{aligned}\right\} \tag{7·24}$$

are squared speeds of principal transverse waves† provided (cf. (7·10) and (7·17))

$$\left.\begin{aligned}(C_1+C_2+2C_3+3C_4+C_5) \neq 0,\\ (C_1+C_2) \neq 0,\\ (C_1+C_2)-C_4 \neq 0.\end{aligned}\right\} \tag{7·25}$$

† Strictly speaking the word *wave* applies only if the corresponding squared speed is strictly positive. Hayes & Rivlin (1961) have pointed out that if a squared speed were negative, then corresponding small disturbances (arising, for example, from Brownian motion) would build up exponentially with time and the material could not exist in the equilibrium state B. In a private communication Dr E. Varley has indicated that vanishing transverse wave speeds may occur at the interface between two solid phases, in an inhomogeneous stress field.

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Further, if $C_4 \neq 0$, amplitude vectors a_B associated with transverse waves which propagate in directions perpendicular to the radial direction (i.e. $N_A J_A = 0$) are in the directions (cf. $(7{\cdot}21)_2$ and (7·22)) J_B and $\epsilon_{BKL} N_K J_L$.

These results indicate that there are, at most, five distinct squared principal wave speeds at every point. An interesting interpretation can be given to the difference between two of them. From $(7{\cdot}24)_1$ and $(7{\cdot}24)_2$,

$$\Sigma - \sigma = D(v_1^2 - v_2^2). \tag{7·26}$$

We recall (cf. (2·2)) that $(\Sigma - \sigma)$ is a quantity which measures the departure from isotropic stress conditions in the equilibrium state B. By (7·26) this stress difference is given directly in terms of the mass density D and the difference between two squared principal transverse wave speeds. In the special case when $\Sigma = \sigma$ we remark that all results obtained in §§ 7 (i) and 7 (ii) remain valid. The only change which occurs if the stress in B is isotropic is a reduction from (at most) three to (at most) two distinct transverse wave speeds.

If it happens that $C_4 = 0$ and $\Sigma = \sigma$, (7·24) with (3·15) and (4·11), shows that

$$W_4 = W_5 = 0$$

and

$$v_1^2 = v_2^2 = v_3^2 = D^{-1}(C_2 + \sigma) = 2D^{-1}(W_1 + W_2).$$

Also, equality of the three squared principal transverse speeds implies that $C_4 = \Sigma - \sigma = 0$. In this case (7·23) yields

$$V_1^2 - V_2^2 = D^{-1}(2C_3 + C_5).$$

Since (cf. (4·11)) C_3 and C_5 depend only on the second derivatives of W (evaluated in B) it follows that the squared principal longitudinal speeds may differ even though $v_1^2 = v_2^2 = v_3^2$. Conversely, if $V_1^2 = V_2^2$, (7·23) yields

$$2C_3 + C_5 + 4C_4 + \Sigma - \sigma = 0,$$

or, with (7·20) and (7·26),

$$2C_3 + C_5 + D\{3v_2^2 + v_1^2 - 4v_3^2\} = 0.$$

From this we see that equality between V_1^2 and V_2^2 does not, in general, imply that the squared principal transverse speeds are equal.

(iv) *Non-principal transverse waves*

In this section we consider waves which propagate with speed V given by $(6{\cdot}8)_1$. Let a_A be the amplitude vector for such a wave. Substitution from $(6{\cdot}8)_1$, $(6{\cdot}9)_1$ and (6·6) into (7·2) yields

$$N_K a_K\{(C_1 + C_2) N_B + (C_3 + C_4)(J_P N_P) J_B\} + J_K a_K\{(C_3 + C_4)(J_P N_P) N_B + [C_4 + C_5(J_P N_P)^2] J_B\} = 0. \tag{7·27}$$

We introduce the quantity Ω defined by

$$\Omega = N_A J_A. \tag{7·28}$$

Since N_A and J_A are unit vectors, Ω satisfies the inequalities

$$-1 \leqslant \Omega \leqslant 1. \tag{7·29}$$

Multiplying (7·27) alternately by N_B and J_B and using (7·28) leads to the system of homogeneous equations

$$\left.\begin{aligned} N_K a_K\{(C_1+C_2)+\Omega^2(C_3+C_4)\}+J_K a_K\{(C_3+2C_4)+\Omega^2 C_5\}\,\Omega = 0, \\ N_K a_K(C_1+C_2+C_3+C_4)\,\Omega+J_K a_K\{C_4+\Omega^2(C_3+C_4+C_5)\} = 0. \end{aligned}\right\} \quad (7·30)$$

Directly from (7·30), the determinant δ of the coefficients of $(N_K a_K)$ and $(J_K a_K)$ is found to be

$$\delta = \{C_4(C_1+C_2)+\Omega^2[C_5(C_1+C_2)-(C_3+C_4)^2]\}\,(1-\Omega^2). \quad (7·31)$$

If $N_A = \pm J_A$, (7·27) reduces to $(7·7)_1$. Then, from the discussion in § 7 (i), we have

$$N_K a_K = 0 \quad \text{when} \quad N_A = \pm J_A. \quad (7·32)$$

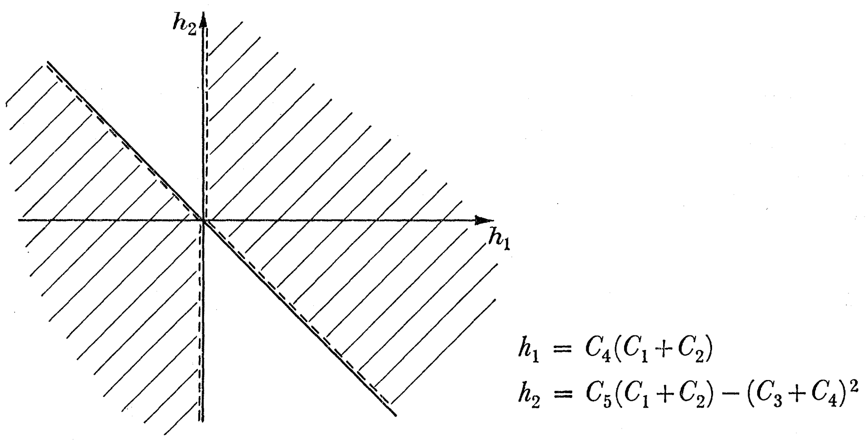

FIGURE 1. Shaded regions of the $h_1 h_2$ plane indicate values of h_1 and h_2 for which $\bar{\delta} \neq 0$ for all Ω^2 in [0, 1].

When $N_A \neq \pm J_A$, (7·28) and (7·29) show that Ω^2 satisfies the condition $0 \leqslant \Omega^2 < 1$. In this case (cf. (7·31)) $\delta = 0$ if and only if $\bar{\delta} = 0$, where

$$\bar{\delta} = C_4(C_1+C_2)+\Omega^2\{C_5(C_1+C_2)-(C_3+C_4)^2\}. \quad (7·33)$$

For the following discussion we define h_1 and h_2 by

$$\left.\begin{aligned} h_1 &= C_4(C_1+C_2), \\ h_2 &= C_5(C_1+C_2)-(C_3+C_4)^2, \end{aligned}\right\} \quad (7·34)$$

and rewrite (7·33) as

$$\bar{\delta} = h_1+\Omega^2 h_2. \quad (7·35)$$

We seek to identify conditions under which $\bar{\delta} \neq 0$ for *all* values of Ω^2 satisfying $0 \leqslant \Omega^2 \leqslant 1$. It is easily seen from (7·35) that $\bar{\delta} \neq 0$ for all Ω^2 in [0, 1] if and only if one of the following four sets of conditions holds: (1) $h_1 > 0$, $h_2 \geqslant 0$; (2) $h_1 > 0$, $h_2 < 0$, $h_1 > |h_2|$; (3) $h_1 < 0$, $h_2 > 0$, $|h_1| > h_2$; or (4) $h_1 < 0$, $h_2 \leqslant 0$. The range of possible values of h_1 and h_2 for which $\bar{\delta} \neq 0$ is indicated by cross-hatching in figure 1. We note that $\bar{\delta} = 0$ only if $h_1 \neq 0$. Since (cf. $(7·17)_1$) $(C_1+C_2) \neq 0$, $(7·34)_1$ shows that $h_1 = 0$ if and only if $C_4 = 0$. Then, from (7·20), $h_1 = 0$ if and only if the two squared principal transverse wave speeds v_2^2 and v_3^2 are equal. Thus, a necessary condition that $\bar{\delta} \neq 0$ is $v_2^2 \neq v_3^2$. As indicated above, sufficient conditions which ensure that $\bar{\delta} \neq 0$ involve h_2 given by $(7{\cdot}34)_2$. Unlike h_1, there is no obvious simple relation by which h_2 can be expressed in terms of principal wave speeds.

Motivated by the discussion in the preceding paragraph, we say the material at radius R in B is *definite transversely isotropic* (d.t.i.) if and only if the quantities $C_4(C_1+C_2)$ and $C_5(C_1+C_2)-(C_3+C_4)^2$, evaluated at R, are such that (cf. (7·33)) $\bar{\delta} \neq 0$ for *all* Ω^2 in $[0, 1]$. In this case the only solution to the system of equations (7·30) is

$$N_K a_K = 0 \quad \text{and} \quad J_K a_K = 0. \tag{7·36}$$

Thus, when the material is d.t.i. $(7.36)_1$ and (7·32), with $(7{\cdot}3)_2$, show that waves governed by the speed $(6{\cdot}8)_1$ are transverse for all directions of propagation. Further, the amplitude vector, a_A, for such waves is perpendicular to the radial direction.

(v) *Other non-principal waves*

In this section we consider waves which propagate with speeds V given by $(6{\cdot}8)_2$. Let a_A^+ and a_A^-, respectively, denote amplitude vectors for waves associated with the (+) and (−) speeds of $(6{\cdot}8)_2$. When it is convenient to consider both types of waves simultaneously, we write $a_A^\pm$.

In the case when $N_A = \pm J_A$, the results of § 7 (i) apply. We simply replace a_A^2 and a_A^3 by a_A^+ and a_A^-, respectively. Thus, from (7·7), with (7·10),

$$\left.\begin{aligned} a_A^+ &= (N_K a_K^+)\, N_A, \\ N_K a_K^- &= 0. \end{aligned}\right\} \tag{7·37}$$

From (6·6) and $(6{\cdot}9)_1$ it follows that

$$Q_{BK} = H_1 \delta_{BK} + (C_1+C_2)\, N_B N_K + (C_3+C_4)\, \Omega (N_B J_K + J_B N_K) + (C_4 + C_5 \Omega^2)\, J_B J_K, \tag{7·38}$$

where Ω is given by (7·28). Substitution from (7·38) into (7·2), with DV^2 given by $(6{\cdot}8)_2$, leads to the equation

$$\begin{aligned} \tfrac{1}{2}\{(H_2-H_1) \pm [(H_2-H_1)^2 - 4H_3]^{\frac{1}{2}}\}\, a_B^\pm = {} & N_B\{(C_1+C_2)\, N_K a_K^\pm + \Omega (C_3+C_4)\, J_K a_K^\pm\} \\ & + J_B\{\Omega (C_3+C_4)\, N_K a_K^\pm + (C_4 + \Omega^2 C_5)\, J_K a_K^\pm\}. \end{aligned} \tag{7·39}$$

Results which apply when $N_B = \pm J_B$ are indicated in (7·37).

When $N_B \neq \pm J_B$, $\epsilon_{BPQ} N_P J_Q$ is a (non-zero) vector perpendicular to the plane of N_B and J_B. Forming the inner product of this vector with (7·39), we obtain

$$\tfrac{1}{2}\{(H_2-H_1) \pm [(H_2-H_1)^2 - 4H_3]^{\frac{1}{2}}\}\, a_B^\pm\, \epsilon_{BPQ} N_P J_Q = 0. \tag{7·40}$$

Thus, providing

$$(H_2-H_1) \pm \{(H_2-H_1)^2 - 4H_3\}^{\frac{1}{2}} \neq 0, \tag{7·41}$$

equation (7·40) implies that a_A^+ and a_A^- lie in the plane of N_A and J_A.

We define h^+ and h^- by

$$\left.\begin{aligned} h^+ &= (H_2-H_1) + \{(H_2-H_1)^2 - 4H_3\}^{\frac{1}{2}}, \\ h^- &= (H_2-H_1) - \{(H_2-H_1)^2 - 4H_3\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{7·42}$$

It will now be shown that $H_3 \neq 0$, if and only if $h^+ \neq 0$ and $h^- \neq 0$. Suppose $h^+ \neq 0$ and $h^- \neq 0$. Then $h^+h^- \neq 0$. From (7·42), $h^+h^- = 4H_3$. Hence, $H_3 \neq 0$. Now suppose $H_3 \neq 0$ and $h^+ = 0$. From $(7{\cdot}42)_1$ we have

$$(H_2 - H_1) = -\{(H_2 - H_1)^2 - 4H_3\}^{\frac{1}{2}},$$

and by squaring both sides of this equation we obtain the contradictory result $H_3 = 0$. A similar contradiction would be obtained by using $(7{\cdot}42)_2$ if we were to assume $h^- = 0$ instead of $h^+ = 0$. Thus, $H_3 \neq 0$ if and only if $h^+ \neq 0$ and $h^- \neq 0$.

From $(6{\cdot}9)_3$ we have

$$H_3 = \{C_4(C_1 + C_2) + \Omega^2[C_5(C_1 + C_2) - (C_3 + C_4)^2]\}(1 - \Omega^2), \tag{7·43}$$

where Ω is given by (7·28). A comparison of (7·43) and (7·31) shows that $H_3 = \delta$. Thus, when $\Omega^2 < 1$ (i.e. when $N_A \neq \pm J_A$), $H_3 \neq 0$ if and only if $\bar{\delta} \neq 0$, where $\bar{\delta}$ is given by (7·33). Using the theorem proved in the preceding paragraph together with (7·40), (7·41) and (7·42) we see that if the material is d.t.i.,

$$a_B^{\pm}\, \epsilon_{BPQ} N_P J_Q = 0 \quad \text{when} \quad N_A \neq \pm J_A. \tag{7·44}$$

In this case (7·44) shows that the amplitude vectors a_A^+ and a_A^- associated with waves which propagate with (+) and (−) speeds, respectively, given by $(6{\cdot}8)_2$ lie in the plane of N_A and J_A, provided that $N_A \neq \pm J_A$.

(vi) *A discussion of non-principal waves*

In § 7 (iv) it has been shown that waves governed by the speed V, where (cf. $(6{\cdot}8)_1$ and $(6{\cdot}9)_1$)

$$V^2 = D^{-1}\{(C_2 + \sigma) + \Omega^2(C_4 + \Sigma - \sigma)\}, \tag{7·45}$$

are transverse for all directions of propagation and that the amplitude vector a_A for such waves is perpendicular to the radial direction if the material is definite transversely isotropic (d.t.i.). We recall that material at radius R is d.t.i. if and only if $\bar{\delta} \neq 0$ for all Ω^2 in [0, 1] where $\bar{\delta}$ is given by (7·33).

In § 7 (v) we have shown that waves governed by the (+) and (−) speeds V given by (cf. $(6{\cdot}8)_2$)

$$V^2 = \tfrac{1}{2}D^{-1}\{H_1 + H_2 \pm [(H_2 - H_1)^2 - 4H_3]^{\frac{1}{2}}\} \tag{7·46}$$

have amplitude vectors a_A^+ and a_A^- which lie in the plane of N_A and J_A (provided $N_A \neq \pm J_A$) if the material is d.t.i. When $N_A = \pm J_A$ (cf. (7·37)) a_A^+ is parallel to and a_A^- is perpendicular to N_A.

It is apparent from (7·39) that for a general direction of propagation N_A, the amplitude vectors a_A^+ and a_A^- are neither parallel to nor perpendicular to N_A. Thus, in contrast to waves governed by the speed (7·45), waves which propagate with speeds (7·46) are, in general, neither longitudinal nor transverse.

8. The motion of transverse wave fronts

In this section we shall derive equations which describe the motion of the transverse waves discussed in § 7 (iv). Combining $(6{\cdot}8)_1$ and $(6{\cdot}9)_1$ yields

$$V^2 = N_P N_Q \left\{\delta_{PQ}\frac{C_2 + \sigma}{D} + J_P J_Q \frac{C_4 + \Sigma - \sigma}{D}\right\}. \tag{8·1}$$

It is apparent from (8·1) that the speed V is a homogeneous function of first degree in the components of the unit normal vector N_A. Thus, the method of description developed in appendix 3 may be used. Substitution from (8·1) into (13·1) yields

$$\left.\begin{aligned}\frac{\mathrm{d}X_A}{\mathrm{d}t} &= \frac{1}{V}\left\{\frac{C_2+\sigma}{D}N_A+\frac{C_4+\Sigma-\sigma}{D}\Omega J_A\right\},\\ \frac{\mathrm{d}N_A}{\mathrm{d}t} &= \frac{1}{2V}(N_A\Omega-J_A)\left\{\left(\frac{C_2+\sigma}{D}\right)'+R^2\Omega^2\left(\frac{C_4+\Sigma-\sigma}{R^2D}\right)'\right\},\end{aligned}\right\} \tag{8·2}$$

where, we recall, $$\Omega = J_A N_A,\quad J_A = X_A/R,\quad R = (X_A X_A)^{\frac{1}{2}}, \tag{8·3}$$

and where we have used the fact that Σ, σ, D and the C_α are functions of R only. From (8·1), with $(8{\cdot}3)_1$, we have

$$V = \pm\left\{\frac{C_2+\sigma}{D}+\Omega^2\frac{C_4+\Sigma-\sigma}{D}\right\}^{\frac{1}{2}}. \tag{8·4}$$

Since V enters the secular equation (6·7) only as V^2, the sign ($\pm$) in (8·4) is not determined. For definiteness the positive sign will be used uniformly throughout.

Appropriate initial conditions for the system (8·2) are of the form

$$X_A(0) = X_A(a,b) \quad\text{and}\quad N_A(0) = N_A(a,b), \tag{8·5}$$

where a and b are suitable parameters which describe the position of the wave front at time $t = 0$.† The solution of the six equations (8·2), subject to initial conditions (8·5), provides a detailed description of the motion of a transverse wave front.

In a sense the information provided by the solution of (8·2) contains too much detail for our purposes, and it will be shown presently that sufficient information can be obtained by considering a reduced system of equations.

We shall first prove that the motion of a point, obtained as a solution to (8·2), lies in a diametral plane of the spherical body.

Let P denote a point on a wave front at the instant $t = 0$. We choose the reference frame X, as may be done without loss of generality, so that the X_1X_2-plane contains the point P and the wave normal at P.‡ In this system $X_3(0) = N_3(0) = 0$ at the point P. We note that $X_3 = N_3 = 0$ satisfies the system (8·2), with $A = 3$, for all times t.

The ray through P is the curve $X_A = X_A(t)$ obtained as the solution to (8·2). Since $X_3(t) \equiv 0$, it follows that the ray through P lies in the X_1X_2-plane. Also since $N_3(t) \equiv 0$, the unit normal to the front at $X_A(t)$ lies in this plane. Then, since the X_1X_2-plane contains the centre of the sphere, it follows that the ray through P lies in a diametral plane.

The above result indicates that it suffices to consider only four of the six equations (e.g. $A = 1, 2$) of the system (8·2) in order to describe a particular ray. Then, by specifying the orientation of X appropriately, the position vector to a point on a ray is of the form, for example, $\{X_1(t), X_2(t), 0\}$ and the unit normal vector at this point is $\{N_1(t), N_2(t), 0\}$. Thus, in general, only two spatial variables (e.g. X_1, X_2) are required to describe a ray.

A further simplification can be made by taking advantage of the underlying spherical and radial symmetry of the equilibrium sphere B.

† Since the right-hand members of (8·2) do not depend upon t explicitly, the choice of the initial instant is arbitrary.

‡ Recall that the origin of X is at the centre of the spherical body.

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Let $X_A(t)$ and $N_A(t)$ be solutions of the system (8.2). Recalling that $J_A = X_A/R$, it follows that the quantities Ω and R in (8·3) are known functions of t. On the other hand, differentiating R and Ω as they appear in (8·3) yields.

$$\left.\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= J_A\frac{\mathrm{d}X_A}{\mathrm{d}t},\\ \frac{\mathrm{d}\Omega}{\mathrm{d}t} &= J_A\frac{\mathrm{d}N_A}{\mathrm{d}t}+\frac{1}{R}N_A\frac{\mathrm{d}X_A}{\mathrm{d}t}-\frac{\Omega}{R}\frac{\mathrm{d}R}{\mathrm{d}t}.\end{aligned}\right\} \tag{8·6}$$

Substitution from (8·2) into (8·6) yields the two equations

$$\left.\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= \frac{\Omega}{V}\frac{C_2+C_4+\Sigma}{D},\\ \frac{\mathrm{d}\Omega}{\mathrm{d}t} &= -\frac{R^2(1-\Omega^2)}{2V}\left\{\left(\frac{C_2+\sigma}{R^2D}\right)'+\Omega^2\left(\frac{C_4+\Sigma-\sigma}{R^2D}\right)'\right\}.\end{aligned}\right\} \tag{8·7}$$

Considering (8·7) as a system of equations for R and Ω†, appropriate initial conditions are $R(0)$, the radial distance to a point P on the wave front at $t = 0$, and $\Omega(0)$, the inclination of the unit normal vector to the radial direction at that point. The solution to (8·7) then gives R and Ω for the point under consideration for subsequent times t. In itself this result does not provide a complete spatial description of a ray, although such a description can be obtained from it.

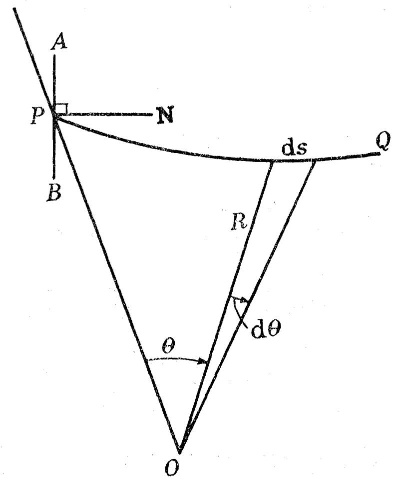

FIGURE 2. The geometry of a ray.

A ray PQ passing through P is shown in figure 2. AB is a portion of the wave front through P. Let θ be the polar angle measured from OP to a general point on the ray and let s denote distance measured along the ray.

We assume that R and Ω are known functions of t which satisfy (8·7) and initial conditions appropriate to the point P. A complete spatial description of the ray is obtained if the dependence of θ on t is specified ($R(t)$ being the other spatial variable).

† We recall that D, Σ, σ, C_2 and C_4 are functions of R and that V is a function of R and Ω given by (8·4).

Since $(\mathrm{d}s/\mathrm{d}t)^2 = (\mathrm{d}X_A/\mathrm{d}t)\,(\mathrm{d}X_A/\mathrm{d}t)$, we have, from $(8\cdot2)_1$,

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{1}{V^2}\left\{\left(\frac{C_2+\sigma}{D}\right)^2 + 2\Omega^2\frac{C_2+\sigma}{D}\frac{C_4+\Sigma-\sigma}{D} + \Omega^2\left(\frac{C_4+\Sigma-\sigma}{D}\right)^2\right\}. \qquad (8\cdot8)$$

Also, the equation for arc length in polar coordinates is

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + R^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2. \qquad (8\cdot9)$$

Eliminating $(\mathrm{d}s/\mathrm{d}t)^2$ from equations (8·8) and (8·9), together with $(8\cdot7)_1$, we find

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \frac{1-\Omega^2}{R^2V^2}\left(\frac{C_2+\sigma}{D}\right)^2. \qquad (8\cdot10)$$

From (8·4) and the fact that $R(t)$ and $\Omega(t)$ are assumed known, we note that V is a known function of t. Then from (8·10)

$$\theta(t) = \int_0^t \frac{C_2+\sigma}{RDV}(1-\Omega^2)^{\frac{1}{2}}\,\mathrm{d}t. \qquad (8\cdot11)$$

Hence, a complete description of rays is obtainable if the solution to the system (8·7) is known.

9. The solution for transverse waves and ray geometry

We introduce the notation

$$C = (C_2+\sigma)/R^2D, \quad c = (C_4+\Sigma-\sigma)/R^2D \qquad (9\cdot1)$$

and rewrite equations (8·7) in the form

$$\left.\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= \frac{\Omega R^2}{V}(C+c),\\ \frac{\mathrm{d}\Omega}{\mathrm{d}t} &= -\frac{R^2(1-\Omega^2)}{2V}(C'+\Omega^2c').\end{aligned}\right\} \qquad (9\cdot2)$$

It is a straight-forward matter to verify that Ω given by

$$\Omega^2 = \frac{\alpha^2-C}{\alpha^2+c}, \qquad (9\cdot3)$$

where α^2 is a constant, satisfies the system of equations (9·2).

From (8·4) and (9·1) we have
$$V^2 = R^2(C+\Omega^2c). \qquad (9\cdot4)$$

Since (cf. (8·3)) Ω is the inner product of two unit vectors, we must have

$$0 \leqslant \Omega^2 \leqslant 1. \qquad (9\cdot5)$$

With (9·5) it follows from (9·4) that V^2 is positive for *all* admissible values of Ω^2 if and only if

$$C > 0 \quad \text{and} \quad C+c > 0. \qquad (9\cdot6)$$

From (7·24) and (9·1) we have

$$v_3^2 = R^2C, \quad v_1^2 = R^2(C+c), \qquad (9\cdot7)$$

and, accordingly, v_3^2 and v_1^2 are positive† if and only if the inequalities (9·6) hold. Henceforth we assume the inequalities (9·6) are valid.

As indicated in (9·5), Ω^2 must satisfy certain inequalities. These inequalities impose restrictions on the value of the integration constant, α^2, appearing in (9·3). We shall now determine these restrictions, basing our argument on the fact that C and c must satisfy (9·6).

If $\Omega^2 = 1$, (9·3) yields $C+c = 0$ if α^2 is finite. This violates $(9{\cdot}6)_2$. Conversely, dividing both numerator and denominator of (9·3) by α^2 and then letting $\alpha^2 \to \infty$, we obtain $\Omega^2 = 1$. Hence, $\Omega^2 = 1$ if and only if $\alpha^2 = \infty$. We need now consider only finite values of α^2 and the inequalities

$$0 \leqslant \Omega^2 < 1. \tag{9·8}$$

A necessary and sufficient condition that Ω^2, given by (9·3), satisfies (9·8) is

$$\alpha^2 \geqslant C. \tag{9·9}$$

To prove this we first assume (9·3) satisfies (9·8) but that (9·9) does not hold. Then $\alpha^2 - C < 0$, and, since $\Omega^2 \geqslant 0$, it follows from (8·3) that $\alpha^2 + c < 0$. Thus, both numerator and denominator in (9·3) are negative. This fact together with the requirement that $\Omega^2 < 1$ yields $C+c < 0$, in violation of (9·6). We conclude that α^2 must satisfy (9·9). Conversely, suppose that (9·9) holds. There are two cases to consider: (1) $c \geqslant 0$; and (2) $c < 0$. In case 1, (9·9) and (9·6) show that $\alpha^2 + c > 0$, $\alpha^2 - C \geqslant 0$ and $\alpha^2 > 0$. These considerations lead to the following sequence of inequalities

$$0 \leqslant \frac{\alpha^2 - C}{\alpha^2 + c} \leqslant \frac{\alpha^2 - C}{\alpha^2} < \frac{\alpha^2}{\alpha^2} = 1,$$

which, with (9·3), is equivalent to (9·8). For case 2 we write $c = -|c|$, and (9·9) and (9·6) yield $\alpha^2 \geqslant C > |c| > 0$. Also, $0 \leqslant \alpha^2 - C < \alpha^2 - |c|$. With these, we have

$$0 \leqslant \frac{\alpha^2 - C}{\alpha^2 - |c|} = \frac{\alpha^2 - C}{\alpha^2 + c} < \frac{\alpha^2 - |c|}{\alpha^2 - |c|} = 1,$$

and once again (9·8) is satisfied. This completes the proof.

From the discussion above, with $\infty > \alpha^2 \geqslant C$, we see that Ω given by (cf. (9·3))

$$\Omega = \pm \left(\frac{\alpha^2 - C}{\alpha^2 + c}\right)^{\frac{1}{2}} \tag{9·10}$$

is a real-valued function which satisfies the inequalities $-1 < \Omega < 1$. From (9·9) and (9·6) we note that $\alpha^2 > 0$. Introducing (9·10) into (9·4) yields

$$V = R\alpha \left(\frac{C+c}{\alpha^2 + c}\right)^{\frac{1}{2}}, \tag{9·11}$$

where we have taken $\alpha = +\sqrt{\alpha^2}$. Substitution from (9·10) and (9·11) into $(9{\cdot}2)_1$ gives the differential equation for R

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \pm \frac{R}{\alpha} (C+c)^{\frac{1}{2}} (\alpha^2 - C)^{\frac{1}{2}}. \tag{9·12}$$

In the particular case when $\alpha^2 = \infty$, we have $\Omega = \pm 1$, and substitution from (9·4) into $(9{\cdot}2)_1$ yields

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \pm R (C+c)^{\frac{1}{2}}. \tag{9·13}$$

† See the footnote in §7 (iii), p. 629.

It is clear that equations (9·12) and (9·13) can be solved for $R = R(t)$ only if the forms of C and c as functions of R are specified. Short of this, some information regarding ray geometry can be obtained from (9·10) and (9·12) directly.

In §8 it was pointed out that the curve $X_A(t)$, obtained as the solution to the system (8·2), is a ray associated with a transverse wave front. Letting U denote the speed of propagation of the front along the ray, we have

$$U = \{(\mathrm{d}X_A/\mathrm{d}t)\,(\mathrm{d}X_A/\mathrm{d}t)\}^{\frac{1}{2}}. \tag{9·14}$$

The unit vector tangent to the ray in the direction of propagation is

$$T_A = U^{-1}\,\mathrm{d}X_A/\mathrm{d}t. \tag{9·15}$$

From $(8{\cdot}2)_1$ and (9·14),

$$U = \frac{1}{V}\left\{\left(\frac{C_2+\sigma}{D}\right)^2 + 2\Omega^2\frac{C_2+\sigma}{D}\,\frac{C_4+\Sigma-\sigma}{D} + \Omega^2\left(\frac{C_4+\Sigma-\sigma}{D}\right)^2\right\}^{\frac{1}{2}}. \tag{9·16}$$

Substitution from (9·10) and (9·11) into (9·16), with (9·1), yields

$$U = (R/\alpha)\,\{\alpha^2(C+c) - Cc\}^{\frac{1}{2}}. \tag{9·17}$$

It is easily shown that the quantity inside the radical in (9·17) is positive when α^2, C and c satisfy (9·6) and (9·9).

A ray is said to be *descending* (*ascending*) if and only if the sign of $J_A T_A$ is strictly negative (positive). From $(8{\cdot}2)_1$ and (9·15), with (9·1), we have

$$J_A T_A = (R^2/VU)\,(C+c)\,\Omega, \tag{9·18}$$

where V and U are given by (9·11) and (9·17), respectively. Since (cf. $(9{\cdot}6)_2$) $(C+c) > 0$, it follows from (9·18) that a ray is descending (ascending) if and only if $\Omega < 0$ ($\Omega > 0$).

Substitution from (9·10) and (9·11) into (8·11), and use of (9·1), yields the expression

$$\theta(t) = \frac{1}{\alpha}\int_0^t C\,\mathrm{d}t. \tag{9·19}$$

From the discussion in §8 we see that the solution $R(t)$ of (9·12) and $\theta(t)$, given by (9·19), together, suffice to determine a ray through any point P of the body B. A particular ray is obtained provided the value of Ω at P is given. Letting $R(0)$ and $\Omega(0)$ denote the values of R and Ω at P, (9·10) yields

$$\alpha = \left\{\frac{C(R(0)) + \Omega^2(0)\,c(R(0))}{1-\Omega^2(0)}\right\}^{\frac{1}{2}}. \tag{9·20}$$

Thus, α is determined in terms of the initial values of R and Ω.

The parameter α, however, has a significance which is not apparent in (9·20). For consider a ray whose deepest point of penetration is to the radius $R = R_L$. At such a 'lowest point' $R(t)$ has a minimum and $\mathrm{d}R/\mathrm{d}t = 0$. Hence, from (9·12),

$$\alpha = \surd C_L, \tag{9·21}$$

where the suffix L indicates that the quantity is evaluated at radius R_L, i.e. $C_L = C(R_L)$. With (9·21), (9·10) shows that $\Omega_L = 0$. Thus, at a lowest point on a ray the wave normal vector N_A is perpendicular to the radial direction J_A. From (9·12), with (9·21) we have

$$(\mathrm{d}^2R/\mathrm{d}t^2)_L = -\tfrac{1}{2}R_L^2(1+c_L/C_L)\,\{C'\}_L. \tag{9·22}$$

From (9·6), it follows that the quantity $(1+c_L/C_L)$ is positive. It then follows from (9·22) that for the particular ray under consideration, $R(t)$ has a true minimum if and only if†

$$\{C'\}_L < 0. \tag{9·23}$$

From $(9·7)_1$, $C = (v_3/R)^2$, and we have $C' = 2(v_3/R)\,(v_3/R)'$. Since v_3/R is positive, a form equivalent to (9·23) is

$$\{(v_3/R)'\}_L < 0. \tag{9·24}$$

Thus, whether $R(t)$ for a particular ray attains a true minimum is determined solely by the behaviour of the principal transverse wave speed v_3. We say that a ray is *regular* if and only if R has a true minimum on the ray.

Consider now the situation where $\mathrm{d}R/\mathrm{d}t = 0$ at some radius R_H, say, and where $\{C'\}_H > 0$. From (9·12) we have, by analogy with (9·21) and (9·22),

$$\left.\begin{aligned}\alpha &= \surd C_H,\\ (\mathrm{d}^2R/\mathrm{d}t^2)_H &< 0.\end{aligned}\right\} \tag{9·25}$$

In this case $R(t)$ has a true maximum and the corresponding ray never penetrates above the 'highest' radius $R = R_H$. With $(9·25)_1$, (9·10) shows that $\Omega_H = 0$. Whether the situation described by $(9·25)_2$ is realized depends only on the behaviour of C.

From the discussion in this section we see that the description of a ray requires that the lower (−) {upper (+)} signs in (9·10) and (9·12) be used on the descending {ascending} portions. Both parts join continuously at the lowest (or highest) points since at such points, $\Omega = 0$ and $\mathrm{d}R/\mathrm{d}t = 0$.

Whereas the behaviour of C determines certain general properties of rays, the form of c governs the orientation of the vectors T_A and N_A relative to the radial direction J_A at each point along a ray. Recalling that $\Omega = J_A N_A$, (9·18), with (9·11) and (9·17), yields

$$(J_A T_A)^2 = \frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)-Cc}\,(J_A N_A)^2. \tag{9·26}$$

At the turning points of a ray, i.e. where $\mathrm{d}R/\mathrm{d}t = 0$, (9·10), (9·12) and (9·18) show that $J_A T_A = J_A N_A = 0$, regardless of the value of c. Let P be a point on a ray other than a turning point. Let C and c in (9·26) be evaluated at P. From (9·26) it follows that the relations

$$(J_A T_A)^2 \gtreqless (J_A N_A)^2 \text{ are equivalent to } \frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)-Cc} \gtreqless 1. \tag{9·27}$$

In turn, we shall prove that the relations

$$\frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)-Cc} \gtreqless 1 \text{ are equivalent to } \quad c \gtreqless 0. \tag{9·28}$$

We note that
$$\alpha^2(C+c)-Cc = (C+c)\,(\alpha^2-C)+C^2 > 0.$$

Thus, the inequalities on the left-hand side of (9·28) imply

$$c(2C+c) \gtreqless 0.$$

† If it should happen that $C' = 0$ at $R = R_L$, it can be shown from (9·12) and (9·21) that $(\mathrm{d}^nR/\mathrm{d}t^n)_L = 0$ for all $n \geqslant 1$ and, hence, that the corresponding solution to (9·12) is $R(t) \equiv R_L$.

Since from (9·6), $2C+c > 0$, this implies $c \gtreqless 0$. Conversely,

$$\text{if} \quad c = 0, \quad \text{then} \quad \frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)-Cc} = 1;$$

$$\text{if} \quad c > 0, \quad \text{then} \quad \frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)-Cc} > \frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)} = \frac{\alpha^2+c}{\alpha^2} > 1;$$

$$\text{if} \quad c < 0, \quad \text{then} \quad \frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)-Cc} < \frac{(\alpha^2+c)\,(C+c)}{\alpha^2(C+c)} = \frac{\alpha^2+c}{\alpha^2} < 1.$$

This completes the proof of the statements in (9·28).

With (9·28), (9·27) yields the result that the relations

$$(J_A T_A)^2 \gtreqless (J_A N_A)^2 \quad \text{are equivalent to} \quad c \gtreqless 0. \tag{9·29}$$

From (9·18) we have that both $J_A T_A$ and $J_A N_A$ $(= \Omega)$ are negative on the descending portion of a ray and positive on the ascending part. This fact, together with (9·29) determines the relative orientation of T_A, N_A and J_A at points along a ray.

From (9·12) we have

$$\mathrm{d}t = \pm \frac{\alpha}{R}\, \frac{\mathrm{d}R}{(C+c)^{\frac{1}{2}}\,(\alpha^2-C)^{\frac{1}{2}}}, \tag{9·30}$$

where $\mathrm{d}t$ is the time required for a disturbance to propagate along a ray between the radii R and $R+\mathrm{d}R$. Let T denote the total time required for propagation along a ray from radius R to the lowest radius R_L and back to R, where $R > R_L$. On the descending (ascending) portion, $\mathrm{d}R$ is negative (positive) and (9·30) yields

$$T = 2\alpha \int_{R_L}^{R} \frac{\mathrm{d}R}{R(C+c)^{\frac{1}{2}}(\alpha^2-C)^{\frac{1}{2}}}. \tag{9·31}$$

Let Δ denote the total angle subtended, relative to the centre of B, by the two points defined by the intersection of the sphere of radius R with the ray described above. From (9·19) and (9·30)

$$\mathrm{d}\theta = \pm \frac{C\,\mathrm{d}R}{R(C+c)^{\frac{1}{2}}\,(\alpha^2-C)^{\frac{1}{2}}}. \tag{9·32}$$

Integration of (9·32) over the interval (R_L, R) yields

$$\Delta = 2 \int_{R_L}^{R} \frac{C\,\mathrm{d}R}{R(C+c)^{\frac{1}{2}}\,(\alpha^2-C)^{\frac{1}{2}}}. \tag{9·33}$$

Since (cf. (9·21)) $\alpha = \surd C_L$, the integrals (9·31) and (9·33) are improper at the lower limit R_L. A sufficient condition which ensures the convergence of these integrals is that C' satisfy (9·23).

Another feature which can be determined from (9·12) and (9·19) is ray curvature. Let P be a point in B. Choose X so that the X_2-axis is along OP. Let $X_1(t)$ and $X_2(t)$ be the coordinates of a point on a ray through P which lies in the X_1X_2-plane. Let κ denote the upward curvature of the ray. Then

$$\kappa = \left(\frac{\mathrm{d}X_1}{\mathrm{d}t}\,\frac{\mathrm{d}^2X_2}{\mathrm{d}t^2} - \frac{\mathrm{d}X_2}{\mathrm{d}t}\,\frac{\mathrm{d}^2X_1}{\mathrm{d}t^2}\right) \Big/ \left\{\left(\frac{\mathrm{d}X_1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}X_2}{\mathrm{d}t}\right)^2\right\}^{\frac{3}{2}}. \tag{9·34}$$

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We introduce polar coordinates (R, θ) according to

$$X_1 = R\sin\theta, \quad X_2 = R\cos\theta. \tag{9·35}$$

Substitution from (9·35) into (9·34) yields

$$\kappa = \left\{R\left(\frac{d\theta}{dt}\frac{d^2R}{dt^2} - \frac{dR}{dt}\frac{d^2\theta}{dt^2}\right) - R^2\left(\frac{d\theta}{dt}\right)^3 - 2\frac{d\theta}{dt}\left(\frac{dR}{dt}\right)^2\right\}\bigg/\left\{\left(\frac{dR}{dt}\right)^2 + R^2\left(\frac{d\theta}{dt}\right)^2\right\}^{\frac{3}{2}}. \tag{9·36}$$

From (9·12) and (9·19) we have

$$\left.\begin{aligned} \frac{dR}{dt} &= \pm\frac{R}{\alpha}(C+c)^{\frac{1}{2}}(\alpha^2-C)^{\frac{1}{2}}, \\ \frac{d^2R}{dt^2} &= \frac{R}{\alpha^2}\{(\alpha^2-C)\,[(C+c)+\tfrac{1}{2}R(C'+c')]-\tfrac{1}{2}R(C+c)\,C'\}, \\ \frac{d\theta}{dt} &= \frac{C}{\alpha}, \\ \frac{d^2\theta}{dt^2} &= \frac{C'}{\alpha}\frac{dR}{dt}. \end{aligned}\right\} \tag{9·37}$$

Substituting from (9·37) into (9·36), we obtain

$$\kappa = \frac{C}{R}\frac{(\alpha^2-C)[\frac{1}{2}R\{c'-C'(1+2c/C)\}-(C+c)]-C[\frac{1}{2}RC'(1+c/C)+C]}{\{\alpha^2(C+c)-Cc\}^{\frac{3}{2}}}. \tag{9·38}$$

We remark that the term in the denominator of (9·38) is real in view of (9·6) and (9·9).

Equation (9·38) gives the curvature at an arbitrary point along a ray in terms of C, c and the parameter α. Making use of (9·21), the curvature at a 'lowest point' of a ray is found to be

$$\kappa_L = -\frac{1}{R_L C_L}\{\tfrac{1}{2}R_L C_L'(1+c_L/C_L)+C_L\}, \tag{9·39}$$

where, from (9·23), $C_L' < 0$. Accordingly, from (9·39) we see that κ_L is positive, zero, or negative as c_L is, respectively, greater than, equal to, or less than the quantity

$$-C_L\left\{\frac{2C_L}{R_L C_L'}+1\right\}.$$

10. Application to seismology theory

In the terminology of seismologists the transverse waves considered in §§7 (iv), 8 and 9 are pure *SH* waves. For such waves, particle motion is horizontal and perpendicular to the propagation direction. It was shown in §9 (cf. (9·24)) that the principal transverse wave speed v_3, given by (7·15), plays a central role in determining whether a ray reaches its greatest depth below the surface at a particular radius. This condition is reminiscent of a similar result from the classical isotropic theory of seismology. In that case the variation of wave speed with radius can be determined from seismological observations directly. So long as a condition corresponding to (9·24) holds, the observable relationship between (cf. (9·31) and (9·33)) T and Δ at the Earth's outer surface is sufficient to determine the variation of wave speed with radius in the isotropic case.

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That the classical method fails to yield correct conclusions if material anisotropy is present was pointed out by Stoneley (1949). In this section we show how it fails for *SH* waves in an anisotropic Earth. We restrict attention to rays whose end-points are on the outer surface of the sphere *B*. We also assume that the relationship between T and Δ is known for waves from surface disturbances which emerge at the outer surface and are identifiable as *SH* fronts.

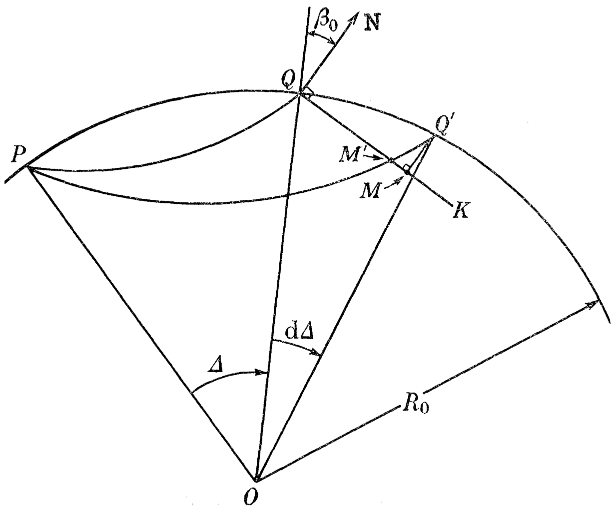

FIGURE 3. The geometry of a ray and wave front at the Earth's surface.

In figure 3 two neighbouring rays, PQ and PQ' are shown. The points P, Q and Q' are on the outer surface of B. An *SH* disturbance is regarded as originating at P. It is detected at Q and Q' at times T and $T+\mathrm{d}T$ later, respectively. Angular distances between P and Q and P and Q' are denoted i and $\Delta+\mathrm{d}\Delta$, respectively. In figure 3, QK denotes the position of the wave front at time T. QK intersects the ray PQ' at point M'. Point M is chosen on QK so that the angle QMQ' is a right angle.

Let β_0 denote the acute angle between J_A and N_A at Q. Then

$$\cos\beta_0 = \Omega_0, \tag{10·1}$$

where the suffix 0 refers to the value at $R = R_0$. From (9·10),

$$\Omega_0 = \left(\frac{\alpha^2-C_0}{\alpha^2+c_0}\right)^{\frac{1}{2}}, \tag{10·2}$$

where α is the ray parameter for the ray PQ. Based on the geometry of figure 3 and the discussion above we have

$$\left.\begin{aligned} MQ'/QQ' &= \sin\beta_0, \\ QQ' &= R_0\,\mathrm{d}\Delta. \end{aligned}\right\} \tag{10·3}$$

Also, since V is the speed of propagation of the wave front (i.e. the phase velocity) we have that the time increment $\mathrm{d}T$ is given by†

$$\mathrm{d}T = MQ'/V_0, \tag{10·4}$$

† An alternate expression for $\mathrm{d}T$ could be written involving the ray velocity U. This is $\mathrm{d}T = M'Q'/U_0$ where (cf. 9·17) $U_0 = \{\alpha^2(C_0+c_0)-C_0c_0\}^{\frac{1}{2}}R_0/\alpha$. It is easily shown that the same final result (10·7) is obtained by considering the geometry of triangle $QM'Q'$ in figure 3.

where, from (9·11),

$$V_0 = R_0\alpha\left(\frac{C_0+c_0}{\alpha^2+c_0}\right)^{\frac{1}{2}}. \tag{10·5}$$

From (10·3) and (10·4) we obtain

$$\frac{dT}{d\Delta} = \frac{R_0}{V_0}\sin\beta_0. \tag{10·6}$$

Noting that $\sin\beta_0 = \{1-\cos^2\beta_0\}^{\frac{1}{2}}$, substitution from (10·1), (10·2) and (10·5) into (10·6) yields

$$\frac{dT}{d\Delta} = \frac{1}{\alpha}. \tag{10·7}$$

Letting R_L denote the radius to the lowest point on ray PQ, (9·21) and (9·7) yield

$$\frac{1}{\alpha} = \frac{R_L}{v_{3L}}. \tag{10·8}$$

From (10·7) and (10·8) it follows that

$$\frac{dT}{d\Delta} = \frac{R_L}{v_{3L}}. \tag{10·9}$$

From classical isotropic theory, the result corresponding to (10·9) is (see, for example, Bullen 1963)

$$\frac{dT}{d\Delta} = \frac{R_L}{v_{sL}}, \tag{10·10}$$

where v_s is the speed of propagation of shear (transverse) waves. Thus, with respect to the propagation of *SH* waves, the role played by the principal transverse speed v_3 is analogous to that of v_s in the isotropic case.

The technique which is generally used to determine the behaviour of wave speed with radius in the isotropic case is the Herglotz–Wiechert method. The starting point in the derivation of this method is the expression for Δ, the total angle subtended by the end-points of a ray. For a ray whose end-points are on the outer surface of the sphere B, (9·33) yields

$$\Delta = 2\int_{R_L}^{R_0}\frac{C\,dR}{R(C+c)^{\frac{1}{2}}(\alpha^2-C)^{\frac{1}{2}}}, \tag{10·11}$$

where α is given by (10·8). Adopting the notation of Bullen (1963) we introduce p and η defined by

$$p = 1/\alpha, \quad \eta = R/v_3. \tag{10·12}$$

From $(9·7)_1$ and $(10·12)_2$

$$\eta = C^{-\frac{1}{2}}. \tag{10·13}$$

Introducing $(10·12)_1$ and (10·13) into (10·11) we obtain

$$\Delta = 2p\int_{R_L}^{R_0}\frac{dR}{R(1+c/C)^{\frac{1}{2}}(\eta^2-p^2)^{\frac{1}{2}}}, \tag{10·14}$$

where, from $(10·12)_2$ and (10·8),

$$p = \eta_L. \tag{10·15}$$

Equation (10·14) is, apart from the factor $(1+c/C)^{\frac{1}{2}}$, identical with Bullen's (1963) equation (7–8). From (9·6), it is seen that this factor is strictly positive for all R. Hence, the only singularity of the integrand occurs when $R = R_L$ (i.e. when $p = \eta$).

An essential requirement in deriving the desired result from (10·14) is that η be a monotone increasing function of R over the interval of integration $[R_L, R_0]$. In particular, this means that

$$\eta'_L = (R/v_3)'_L > 0. \tag{10·16}$$

Since $(R/v_3)' = -(R/v_3)^2 (v_3/R)'$, (10·16) is equivalent to (9·24).

Details of the steps involved in proceeding from (10·14) to the final expression

$$\int_0^{\hat{\Delta}} \cosh^{-1}(p/\hat{p})\, \mathrm{d}\Delta = \pi \int_{\hat{R}}^{R_0} \frac{\mathrm{d}R}{R} (1+c/C)^{-\frac{1}{2}} \tag{10·17}$$

are entirely analogous to those given by Bullen (1963, Chapter 7), and will not be described here. In (10·17) $\hat{\Delta}$ is a fixed angle, $\hat{R}$ is the radius to the lowest point on a ray which subtends a total angle $\hat{\Delta}$, p is a known function of Δ given by (cf. (10·7) and (10·12))

$$p = \mathrm{d}T/\mathrm{d}\Delta, \tag{10·18}$$

and $\hat{p} = p(\hat{\Delta})$.

In the classical isotropic case, the expression corresponding to (10·17) is (Bullen (1963))

$$\int_0^{\hat{\Delta}} \cosh^{-1}(p/\hat{p})\, \mathrm{d}\Delta = \pi \int_{\hat{R}}^{R_0} \frac{\mathrm{d}R}{R}, \tag{10·19}$$

in which $\hat{\Delta}$, $\hat{R}$, p and $\hat{p}$ have precisely the same meanings as above.

With (10·18), the left members of (10·17) and (10·19) can be evaluated. The result is a known number $k(\hat{\Delta})$, say, where

$$k(\hat{\Delta}) = \int_0^{\hat{\Delta}} \cosh^{-1}(p/\hat{p})\, \mathrm{d}\Delta. \tag{10·20}$$

If the material is isotropic so that (10·19) applies, we obtain, with (10·20),

$$k(\hat{\Delta}) = \pi \ln (R_0/\hat{R}). \tag{10·21}$$

From (10·21)

$$\hat{R} = R_0 \exp\{-k(\hat{\Delta})/\pi\}, \tag{10·22}$$

and from (10·10),

$$v_s(\hat{R}) = \frac{\hat{R}}{(\mathrm{d}T/\mathrm{d}\Delta)_{\Delta=\hat{\Delta}}}. \tag{10·23}$$

With $\hat{R}$ given by (10·22), equation (10·23) yields the value of v_s at $R = \hat{R}$. In this way, by choosing various values of $\hat{\Delta}$ in (10·20), the variation of v_s with R can be established in the isotropic case.

From (9·7) we have

$$(1+c/C)^{-\frac{1}{2}} = v_3/v_1, \tag{10·24}$$

where, from (7·24),

$$\left.\begin{aligned} v_3 &= \{(C_2+\sigma)/D\}^{\frac{1}{2}}, \\ v_1 &= \{(C_2+C_4+\Sigma)/D\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{10·25}$$

With (10·24) and (10·20), (10·17) becomes

$$k(\hat{\Delta}) = \pi \int_{\hat{R}}^{R_0} \frac{\mathrm{d}R}{R} (v_3/v_1). \tag{10·26}$$

Equation (10·26) indicates the manner in which the classical Herglotz–Wiechert method breaks down if account is taken of material anisotropy of the Earth. For unless $v_1 = v_3$, the result of integrating the right-hand number will not be $\pi \ln (R_0/\hat{R})$ as in (10·21).

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In a special case it may happen that C_4, Σ and σ are related by $C_4+\Sigma = \sigma$. Then (10·25) shows that $v_1 = v_3$ and the isotropic results (10·21), (10·22) and (10·23) apply. Under these conditions the material appears isotropic with respect to the propagation of *SH* waves.

11. Appendix 1. The strain-energy function

Let P be a generic material point of the body B and let W^P denote the strain energy at P, measured per unit volume of B.

In figure 4 is shown the fixed frame X and the particle P in B. The vector OP is in the radial direction at P. We introduce an auxiliary rectangular frame $\bar{X}$, with origin at O, such that the $\bar{X}_1$ axis is in the direction OP, the orientation of $\bar{X}$ being otherwise arbitrary. Let $\bar{G}_{KL}$ denote the components of the Cauchy–Green strain tensor relative to $\bar{X}$ for an arbitrary deformation. For an elastic material, we have

$$W^P = W^P(\bar{G}_{KL}). \tag{11·1}$$

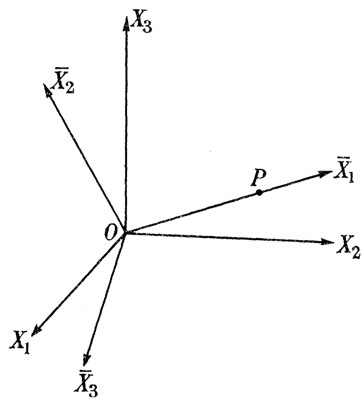

Figure 4. The reference frame X and an auxiliary frame $\bar{X}$.

The statement that the material at P is transversely isotropic with respect to the radial direction means that W^P in (11·1) must remain invariant under all transformations of $\bar{X}$ which leave the $\bar{X}_1$ direction unchanged. This condition leads to the requirement (Ericksen & Rivlin (1954), Adkins (1955)) that W^P depends on the components of $\bar{G}_{KL}$ only through the five quantities I_α ($\alpha = 1, \ldots, 5$) defined by

$$I_1 = \bar{G}_{KK}, \quad I_2 = \tfrac{1}{2}(\bar{G}_{KK}\bar{G}_{LL}-\bar{G}_{KL}\bar{G}_{LK}), \quad I_3 = |\bar{G}_{KL}|, \quad I_4 = \bar{G}_{11}, \quad I_5 = \bar{G}_{1K}\bar{G}_{K1}. \tag{11·2}$$

Let a_{KM} denote the cosine of the angle between the $\bar{X}_K$ and X_M axes. Then, a_{KM} satisfies the orthogonality conditions

$$a_{KM}a_{LM} = a_{MK}a_{ML} = \delta_{KL}, \quad |a_{KL}| = \pm 1. \tag{11·3}$$

Denoting the components of the Cauchy–Green strain tensor in the fixed frame X by G_{KL}, we have

$$\bar{G}_{KL} = a_{KM}a_{LN}G_{MN}. \tag{11·4}$$

We note, from the definition of a_{KM} and J_M (see §2) that

$$a_{1M} = \pm J_M. \tag{11·5}$$

Introducing the relations (11·4) into (11·2) and employing (11·3) and (11·5), we obtain

$$\left.\begin{aligned} I_1 = G_{KK}, \quad I_2 = \tfrac{1}{2}(G_{KK}G_{LL} - G_{KL}G_{LK}), \quad I_3 = |G_{KL}|, \\ I_4 = J_K J_L G_{KL}, \quad I_5 = J_K J_L G_{KM} G_{ML}. \end{aligned}\right\} \tag{11·6}$$

Since the body has spherical symmetry in its reference configuration B, the dependence of the strain-energy function on P is replaced by an explicit dependence on the radial distance R. Thus, in place of (11·1), we may write

$$W^P = W(I_\alpha, R) \quad (\alpha = 1, \ldots, 5)$$

where the I_α are given by (11·6).

12. Appendix 2. Some general geometrical and kinematical results

Since the method we employ, in this paper, to describe the propagation of waves is *material* rather than *spatial*, it seems worthwhile to include a discussion showing the connection between the two. The results derived in this appendix are not new. They can be obtained directly from the general equations presented by Truesdell & Toupin (1960).

Let X_A denote material coordinates, measured in a fixed rectangular frame X. Let $x_i = x_i(X_A, t)$ be a motion, where x_i are coordinates measured in X. Let Γ_t and γ_t be the material and spatial descriptions, respectively, of a wave; i.e.

$$\left.\begin{aligned} \Gamma_t &= \{X_A | \Phi(X_A, t) = 0\}, \\ \gamma_t &= \{x_i | \phi(x_i, t) = 0\}, \end{aligned}\right\} \tag{12·1}$$

where Φ and ϕ are related by

$$\Phi(X_A, t) \equiv \phi(x_i(X_A, t), t). \tag{12·2}$$

The material unit normal vector and speed of propagation associated with Γ_t are, respectively,

$$\left.\begin{aligned} N_A &= \Phi_{,A}/(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}}, \\ V &= -\dot{\Phi}/(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}}. \end{aligned}\right\} \tag{12·3}$$

The spatial unit normal vector and speed of displacement associated with γ_t are, respectively,

$$\left.\begin{aligned} n_i &= \frac{\partial \phi}{\partial x_i} \bigg/ \left(\frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_k}\right)^{\frac{1}{2}}, \\ u &= -\dot{\phi} \bigg/ \left(\frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_k}\right)^{\frac{1}{2}}. \end{aligned}\right\} \tag{12·4}$$

We differentiate (12·2) with respect to t and obtain

$$\dot{\Phi} = \frac{\partial \phi}{\partial x_i}\dot{x}_i + \dot{\phi}. \tag{12·5}$$

With the aid of $(12·3)_2$ and (12·4), we can write (12·5) in the form

$$(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}} V = \left(\frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_k}\right)^{\frac{1}{2}} (u - \dot{x}_i n_i). \tag{12·6}$$

The quantity $(u-\dot{x}_i n_i)$ is the local speed of propagation. We set

$$v = u-\dot{x}_i n_i \tag{12·7}$$

and rewrite (12·6) as

$$(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}} V = \left(\frac{\partial\phi}{\partial x_k}\frac{\partial\phi}{\partial x_k}\right)^{\frac{1}{2}} v. \tag{12·8}$$

Differentiation of (12·2) with respect to X_A, together with $(12·3)_1$ and $(12·4)_1$, yields

$$N_A(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}} = n_i x_{i,A}\left(\frac{\partial\phi}{\partial x_k}\frac{\partial\phi}{\partial x_k}\right)^{\frac{1}{2}}. \tag{12·9}$$

From (12·8) and (12·9) there follows

$$N_A/V = n_i x_{i,A}/v. \tag{12·10}$$

Equation (12·10) gives, in compact form, the connexion between various geometrical and kinematical quantities associated with material and spatial descriptions of a wave.

If the deformation $x_i(X_A, t)$ is regarded as a small deformation from the reference configuration, we write, as in § 4,

$$x_B = X_B+\epsilon u_B(X_K, t), \tag{12·11}$$

where ϵ is a small constant. As above we let $\Phi = 0$ and $\phi = 0$ be the material and spatial descriptions of a wave. From (12·11) we have

$$\partial X_A/\partial x_B = \delta_{AB}-\epsilon u_{A,B}. \tag{12·12}$$

Differentiating the equation $\Phi(X_A, t) \equiv \phi(x_A, t)$ with respect to x_B and using (12·12) yields

$$\partial\phi/\partial x_B = \Phi_{,B}-\epsilon\Phi_{,A}u_{A,B}. \tag{12·13}$$

With the aid of $(12·3)_1$ and $(12·4)_1$, we obtain from (12·13) that

$$n_B = \{N_B-\epsilon N_A u_{A,B}\}(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}}\Big/\left(\frac{\partial\phi}{\partial x_p}\frac{\partial\phi}{\partial x_p}\right)^{\frac{1}{2}}. \tag{12·14}$$

Since n_B and N_B are unit vectors, we have, from (12·14),

$$\left\{\Phi_{,K}\Phi_{,K}\Big/\frac{\partial\phi}{\partial x_p}\frac{\partial\phi}{\partial x_p}\right\}^{\frac{1}{2}} = 1+\epsilon N_A N_B u_{A,B}. \tag{12·15}$$

With the aid of (12·15) we obtain from (12·14)

$$n_B = N_B+\epsilon\{N_B N_K N_L u_{K,L}-N_K u_{K,B}\}. \tag{12·16}$$

In place of (12·2) we write (cf. (12·11))

$$\Theta(X_A, t) = \phi(X_A+\epsilon u_A(X_K, t), t). \tag{12·17}$$

Differentiating (12·17) with respect to t yields

$$\dot{\Phi} = \frac{\partial\phi}{\partial x_A}\epsilon\dot{u}_A+\dot{\Phi}. \tag{12·18}$$

The linearized counterpart of (12·6) is obtained from (12·18) and (12·16). Thus,

$$V(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}} = \left(\frac{\partial\phi}{\partial x_p}\frac{\partial\phi}{\partial x_p}\right)^{\frac{1}{2}}(u-\epsilon N_B\dot{u}_B), \tag{12·19}$$

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where V and u have the same meanings as before. That is, V is the material speed of propagation and u is the spatial speed of displacement of the wave. As in (12·7),

$$v = u - \epsilon \ddot{u}_B N_B \tag{12·20}$$

is the local speed of propagation. From (12·15) and (12·19), with (12·20), we have

$$\left.\begin{aligned} V\{1+\epsilon N_A N_B u_{A,B}\} &= v, \\ \text{or} \quad V &= \{1-\epsilon N_A N_B u_{A,B}\} v. \end{aligned}\right\} \tag{12·21}$$

Equations (12·21) may be derived directly from (12·10) by using (12·11) and (12·16). From (12·21) we note that the material and local speeds of propagation differ by a term of order ϵ for small deformations.

13. Appendix 3. The material description of seismic waves

Our aim in this section is to derive a set of ordinary differential equations which describes the propagation of seismic waves. These equations are

$$\left.\begin{aligned} \frac{dX_A}{dt} &= \frac{\partial V}{\partial N_A}, \\ \frac{dN_A}{dt} &= (N_A N_B - \delta_{AB}) \frac{\partial V}{\partial X_B}, \end{aligned}\right\} \tag{13·1}$$

where V is the speed of propagation of a wave whose material unit normal vector is N_A.

Before taking up this matter we shall show that the direction of energy propagation (i.e. the ray direction) is parallel to the vector $\partial V/\partial N_A$. As in § 5 let $\Phi(X_A, t) = 0$ be the material description of an isolated acceleration wave. At a fixed particle X_A we assume $\dot{\Phi} \neq 0$ for all t. Then, the equation $\Phi = 0$ may be solved in the form

$$\Phi(X_A, t) = \Psi(X_A) - t = 0. \tag{13·2}$$

With the aid of (13·2), equations (5·1) and (5·2) become

$$\left.\begin{aligned} N_A &= \Psi_{,A}/(\Psi_{,K}\Psi_{,K})^{\frac{1}{2}}, \\ V &= 1/(\Psi_{,K}\Psi_{,K})^{\frac{1}{2}}. \end{aligned}\right\} \tag{13·3}$$

We assume the displacement u_A of § 4 is given in the form of a 'generalized wave', namely,

$$u_A = b_A f(\phi), \tag{13·4}$$

where

$$\left.\begin{aligned} b_A &= b_A(X_K), \\ \phi &= \Psi(X_K) - t, \end{aligned}\right\} \tag{13·5}$$

and where b_A and its derivatives are continuous, f and its first derivative are continuous and the second derivative of f is continuous except at $\phi = 0$; i.e.

$$\lim_{\delta\to 0} f''(\delta) \neq \lim_{\delta \to 0} f''(-\delta), \tag{13·6}$$

where a prime denotes differentiation with respect to the entire argument. We denote the discontinuity of f'' expressed in (13·6) by $[f'']$. From (13·4) and (13·5), with (13·3), it follows that

$$\left.\begin{aligned} [u_{K,LA}] &= [f'']\, b_K N_L N_A / V^2, \\ [\ddot{u}_B] &= [f'']\, b_B. \end{aligned}\right\} \tag{13·7}$$

Substitution from (13·7) into (6·3) yields

$$(S_{AL}\delta_{BK}+C_{ABKL})\,[f'']\,b_K N_L N_A/V^2 = D[f'']\,b_B,$$

or, since $[f''] \neq 0$ at the discontinuity,

$$(S_{AL}\delta_{BK}+C_{ABKL})\,b_K N_L N_A = DV^2 b_B. \tag{13·8}$$

We multiply both sides of (13·8) by b_B to obtain

$$DV^2 b_B b_B = b_B b_K (S_{AL}\delta_{BK}+C_{ABKL})\,N_L N_A. \tag{13·9}$$

Regarding (13·9) as a relation between V and N_A, at a fixed particle X_A, differentiation of this equation with respect to N_A yields

$$\frac{\partial V}{\partial N_A} = \frac{1}{DV}\frac{b_B b_K}{(b_M b_M)}(S_{AL}\delta_{BK}+C_{ABKL})\,N_L. \tag{13·10}$$

Also from (13·4) and (13·5), with (13·3), it follows that

$$\left.\begin{aligned} u_{K,L} &= b_{K,L} f(\phi) + b_K f'(\phi)\,N_L/V,\\ \dot{u}_B &= -b_B f'. \end{aligned}\right\} \tag{13·11}$$

Substitution from (13·11) into (4·18) yields

$$\begin{aligned}\tilde{p}_A = \epsilon s_{AB} b_B f' + \epsilon^2 (S_{AL}\delta_{BK}+C_{ABKL})\,b_{K,L} b_B f f' \\ + \epsilon^2 (S_{AL}\delta_{BK}+C_{ABKL})\,b_B b_K (N_L/V)\,(f')^2.\end{aligned} \tag{13·12}$$

This is the energy flux vector for the deformation (13·4). Noting that $f' = -\mathrm{d}f/\mathrm{d}t$, integration of (13·12) with respect to t, holding X_A constant, yields

$$\int_{-\infty}^{\infty} \tilde{p}_A\,\mathrm{d}t = \epsilon^2 (S_{AL}\delta_{BK}+C_{ABKL})\,b_B b_K (N_L/V) \int_{-\infty}^{\infty} (f')^2\,\mathrm{d}t, \tag{13·13}$$

where we assume that $f \to 0$ as $t \to \pm\infty$. Equation (13·13) gives the direction of energy propagation. A comparison of (13·13) and (13·10) shows that $\partial V/\partial N_A$ is in this direction.

We now turn to the problem of deriving the system of equations (13·1). In § 6 it has been shown that the speed of propagation of a seismic wave is given by one of equations (6·8). From (6·8) and (6·9) we see that this speed, V, may be written as a homogeneous function of the first degree in the components of the unit normal vector N_A. Also, V depends on the material coordinates X_A. To express this dependence and homogeneity explicitly, we write

$$V = \bar{V}(N_A, X_A), \tag{13·14}$$

where $\bar{V}$ satisfies the Euler condition

$$\bar{V} = N_A(\partial \bar{V}/\partial N_A). \tag{13·15}$$

Letting $\Phi(X_A, t) = 0$ denote the wave front, we define the functions $\bar{N}_A$ and $\bar{U}$ by (cf. (5·1) and (5·2))

$$\left.\begin{aligned} \bar{N}_A(X_B,t) &= \Phi_{,A}/(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}},\\ \bar{U}(X_B,t) &= -\dot{\Phi}/(\Phi_{,K}\Phi_{,K})^{\frac{1}{2}}. \end{aligned}\right\} \tag{13·16}$$

On $\Phi = 0$ we have, of course,

$$\left.\begin{aligned} N_A &= \bar{N}_A,\\ V = \bar{U} &= \bar{V}. \end{aligned}\right\} \tag{13·17}$$

Consider a point $X_A(t)$ which moves with velocity $\partial\overline{V}/\partial N_A$. That is,

$$\frac{\mathrm{d}X_A}{\mathrm{d}t} = \frac{\partial\overline{V}}{\partial N_A}(\overline{N}_B(X_K,t), X_K). \tag{13·18}$$

From (13·15), (13·17) and (13·18), we obtain

$$N_A\frac{\mathrm{d}X_A}{\mathrm{d}t} = V, \tag{13·19}$$

so the point $X_A(t)$ remains on the wavefront. In the following we assume all functions which depend on (X_A, t) are evaluated at the argument $(X_A(t), t)$ where $X_A(t)$ satisfies (13·18). In particular, for any function $f(X_K, t)$ we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = f_{,K}\frac{\mathrm{d}X_K}{\mathrm{d}t} + \dot{f}. \tag{13·20}$$

Differentiation of $(13·16)_1$ with respect to t and use of $(13·17)_1$ leads to the expression

$$\frac{\mathrm{d}N_A}{\mathrm{d}t} = (\delta_{AB} - N_A N_B)\frac{1}{(\Phi_{,P}\Phi_{,P})^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{,B}. \tag{13·21}$$

It will now be shown that

$$\frac{1}{(\Phi_{,P}\Phi_{,P})^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{,B} = -\overline{V}_{,B}. \tag{13·22}$$

From (13·16) and (13·17),

$$\left.\begin{aligned}\overline{N}_{K,B} &= (\Phi_{,KB} - N_K N_L \Phi_{,LB})/(\Phi_{,P}\Phi_{,P})^{\frac{1}{2}},\\ \overline{U}_{,B} &= -(\dot{\Phi}_{,B} + V N_L \Phi_{,LB})/(\Phi_{,P}\Phi_{,P})^{\frac{1}{2}}.\end{aligned}\right\} \tag{13·23}$$

From (13·14), (13·16) and (13·17), we have

$$\overline{U}(X_B, t) = \overline{V}(\overline{N}_A(X_B, t), X_A). \tag{13·24}$$

Differentiation of (13·24) and use of (13·18) yields

$$\overline{U}_{,B} = \overline{N}_{K,B}(\mathrm{d}X_K/\mathrm{d}t) + \overline{V}_{,B}. \tag{13·25}$$

We eliminate $\overline{U}_{,B}$ from $(13·23)_2$ and (13·25), make use of $(13·23)_1$ and (13·19), and obtain

$$(\Phi_{,KB}(\mathrm{d}X_K/\mathrm{d}t) + \dot{\Phi}_{,B})/(\Phi_{,P}\Phi_{,P})^{\frac{1}{2}} = -\overline{V}_{,B}. \tag{13·26}$$

From (13·20)

$$\frac{\mathrm{d}\Phi_{,B}}{\mathrm{d}t} = \Phi_{,KB}\frac{\mathrm{d}X_K}{\mathrm{d}t} + \dot{\Phi}_{,B}. \tag{13·27}$$

The result (13·22) follows from (13·26) and (13·27).

Substitution from (13·22) into (13·21) yields

$$\mathrm{d}N_A/\mathrm{d}t = -(\delta_{AB} - N_A N_B)\overline{V}_{,B}, \tag{13·28}$$

By regarding the function $\overline{V}$ appearing in (13·14) as being evaluated at the argument $X_A(t)$, $N_A(t)$, where X_A and N_A satisfy (13·18) and (13·28), we obtain the system of equations (13·1). By the manner in which these equations were derived, the point $X_A(t)$ moves with the front and at this point the unit normal vector is $N_A(t)$. The velocity $\mathrm{d}X_A/\mathrm{d}t$ is the ray velocity. The solution $X_A(t)$ to (13·1) is the corresponding ray.

14. Appendix 4. The case of isotropic initial stress

In the traditional treatment of many seismological problems it is customary to assume the equilibrium state of stress is a hydrostatic pressure. For this reason it seems worthwhile to include the specialization of certain results obtained in the main body of this paper to cases where the initial stress is isotropic.

If the initial stress is isotropic, we replace equation (2·2) by

$$S_{AB} = S(R)\,\delta_{AB}, \tag{14·1}$$

and note that the results of § 2 apply provided we set

$$\Sigma = \sigma = S \quad \text{(say)}. \tag{14·2}$$

Equations (2·8) and (2·10) are replaced by

$$\left.\begin{aligned} S' &= DG,\\ S(R_0) &= 0, \end{aligned}\right\} \tag{14·3}$$

respectively.

If the material of the equilibrium sphere is transversely isotropic with respect to the radial direction, then just as in § 3 the strain energy function W depends on (cf. (3·8)), $I_1, \ldots, I_5$ and upon R. The dependence of W upon the I_α is restricted by the conditions (cf. (3·15))

$$\left.\begin{aligned} S &= 2(W_1+2W_2+W_3),\\ 0 &= 2(W_4+2W_5). \end{aligned}\right\} \tag{14·4}$$

By making the substitution (14·2) in all equations of §§ 4–10, results which apply to the present case are obtained directly. From (7·23) and (7·24) it follows that there are at most four distinct principal wave speeds given by

$$\left.\begin{aligned} V_1^2 &= D^{-1}(C_1+2C_2+2C_3+4C_4+C_5+S),\\ V_2^2 &= D^{-1}(C_1+2C_2+S),\\ v_1^2 = v_2^2 &= D^{-1}(C_2+C_4+S),\\ v_3^2 &= D^{-1}(C_2+S), \end{aligned}\right\} \tag{14·5}$$

where the C_α are given by (4·11).

As in (9·1) we introduce $\overline{C}$ and $\bar{c}$ defined by

$$\overline{C} = (C_2+S)/R^2D, \quad \bar{c} = C_4/R^2D. \tag{14·6}$$

The equations for R and Ω are identical in form with (9·2) with C and c replaced by $\overline{C}$ and $\bar{c}$, respectively. In this case (cf. (9·3))

$$\Omega^2 = \frac{\alpha^2-\overline{C}}{\alpha^2+\bar{c}}, \tag{14·7}$$

where α^2 must satisfy (cf. (9·9)) $\qquad \alpha^2 \geqslant \overline{C}. \qquad$ (14·8)

The equation corresponding to (9·12) is

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \pm\frac{R}{\alpha}(\overline{C}+\bar{c})^{\frac{1}{2}}(\alpha^2-\overline{C})^{\frac{1}{2}}. \tag{14·9}$$

The discussion in § 9 regarding ray geometry applies here also. The behaviour of $\bar{C}$ governs the depth of penetration of a ray while the sign of $\bar{c}$ determines the relative orientation of the ray tangent vector and the wave normal vector with respect to the radial direction.

Results of § 10 regarding the use of seismographical $T-\Delta$ data follows from (10·26). Thus,

$$k(\hat{\Delta}) = \pi \int_{\hat{R}}^{R_0} \frac{\mathrm{d}R}{R} \frac{v_3}{v_1}, \tag{14·10}$$

where (cf. (14·5))

$$\left.\begin{aligned} v_3 &= \{(C_2+S)/D\}^{\frac{1}{2}}, \\ v_1 &= \{(C_2+C_4+S)/D\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{14·11}$$

It follows from (14·10) that the condition of isotropic initial stress does not remove the difficulty in applying the classical Herglotz–Wiechert method. According to (14·11), the right-hand member of (14·10) reduces to $\pi \ln (R_0/\hat{R})$ only if $C_4 = 0$, i.e. only if $v_1 = v_3$. In this regard see the discussion in § 7 (iv).

A comparison of the selected results shown in (14·4)–(14·11) with those of §§ 5–10 indicates that the only significant simplification arising from the assumption of isotropic initial stress is in the reduction from five to four principal wave speeds. Qualitative aspects of the propagation of *SH* waves are unchanged.

We finally point out that the case where material response is isotropic is included. Under this condition W depends only upon I_1, I_2 and I_3 given by (3·7) and, of course, upon R. That is,

$$W = W(I_1, I_2, I_3, R). \tag{14·12}$$

The dependence expressed by (14·12) is restricted by the requirement (cf. (14·4))

$$S = 2(W_1 + 2W_2 + W_3), \tag{14·13}$$

where

$$W_\alpha = \frac{\partial W}{\partial I_\alpha}(3, 3, 1, R). \tag{14·14}$$

Results can be obtained immediately for the isotropic material by setting

$$\partial W/\partial I_4 = \partial W/\partial I_5 = 0 \quad \text{and} \quad \Sigma = \sigma = S$$

in §§ 3–6. In particular, from (4·11), we have

$$C_3 = C_4 = C_5 = 0. \tag{14·15}$$

Thus, from (6·6), the linear isotropic acoustic tensor is

$$Q_{BK} = (C_2+S)\,\delta_{BK} + (C_1+C_2)\,N_B N_K. \tag{14·16}$$

From (14·16) follows the well-known result that there are two squared wave speeds,

$$\left.\begin{aligned} v_p^2 &= D^{-1}(C_1+2C_2+S), \\ v_s^2 &= D^{-1}(C_2+S). \end{aligned}\right\} \tag{14·17}$$

The analysis of the motion of wave fronts and their associated rays given in §§ 8 and 9 applies to the completely isotropic case. Thus, for *SH* waves we have from (9·10) and (9·12), with (9·1), (14·2), (14·15) and (14·17),

$$\left.\begin{aligned} \Omega &= \pm (1/\alpha)\,(\alpha^2 - v_s^2/R^2)^{\frac{1}{2}}, \\ \mathrm{d}R/\mathrm{d}t &= \pm (1/\alpha)\,v_s(\alpha^2 - v_s^2/R^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{14·18}$$

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